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MULTIPLIERS ON COMMUTATIVE FRÉCHET ALGEBRAS

BY

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"Non of the work referred to in this thesis has been submitted in support of an application for another degree to this or any other university or institution of learning."

Areej Saleh Salem AL-harbi

TESTIMONY

This is to certify that the candidate Areej Saleh Salem AL-harbi worked under the supervision of the undersigned and completed this thesis to meet the partial requirement of a degree of Master of Science in Mathematics

Dr. Liaqat Ali Khan
(Supervisor)

ABSTRACT

In this thesis, we have studied both algebraic and topological properties of multipliers. A multiplier on an algebra A is a mapping $T : A \to A$ such that $T(a)b = aT(b)$ for all $a, b \in A$. We consider conditions on a Banach or topological algebra A (such as A is faithful or has an approximate identity) to ensure the linearity and continuity of T. We also consider the problems of embedding of A into the algebras of multipliers $M_l(A), M_r(A)$ and $M(A)$ as (left, right) ideal. We study completeness and dense properties of these algebras under the uniform and strict topologies. We give various characterizations of the multiplier algebra $M(A)$, as follows: If A is a commutative algebra with identity, then $M(A)$ is algebraically isomorphic with the algebra $\{L_x : x \in A\}$ of left regular representations. If A is a commutative Banach algebra (or more generally, a Fréchet algebra), then each $T \in M(A)$ is determined by a continuous complex-valued function on the maximal ideal space $\Delta(A)$ of A which is bounded if A is a semisimple Banach algebra. In particular, if $A = C_0(X)$ with X a locally compact Hausdorff space, then $M(A) = C_b(X)$. Further, a linear mapping $T : A \to A$ is a multiplier $\Leftrightarrow$ it is a "Jordan multiplier". We also study various properties of the maximal ideal space $\Delta(M(A))$ of the multiplier algebras $M(A)$ with respect to both the Gelfand and hull-kernel topologies. The set of isometric multipliers need not be a vector space; however, it forms a Hausdorff topological group with respect to the strict topology and is compact under some additional assumptions. When A is a topological algebra with an orthogonal Schauder basis, $M(A)$ can be continuously embedded into the algebra s of sequences of complex numbers; if, in addition, A has an identity and an unconditional orthogonal basis, then $M(A) \cong l_\infty$. The above results include, among others, the work of Wang (1961), Birtel (1961,1962), Johnson (1964, 1966), Máté (1965), Busby (1968) and Larsen (1971) in the setting of Banach algebras and the more recent work of Husain (1989) and Khan-Mohammed-Thaheem (1999) in the general setting of Fréchet algebras.

مستخلص

درسنا في هذه الرسالة ، كل الخواص الجبرية والتبولوجية للمضـــاعفات. المضـــاعف على الجبر A هو تطبيق

$$T : A \to A$$

بحيث أن $T(a)b = aT(b)$

لكل $a,b \in A$. و بحثنا عدة شروط على الجبر الباناخ أو الجبر التبولوجي A (مثلاً عندما تكون A محببة أو أن A لها عنصر محايد تقريبي) لنضمن أن تكون T خطيـــة ومتصلة. وأيضا بحثنا مسائل دمج A داخل جبور المضــــاعفات $M_l(A), M_r(A), M(A)$ كجزء مثالي أيسر وجزء مثالي أيمن . ودرسنا خواص الاكتمال والكثافة لهذه الجبـــور في ظل التبولوجيات المنتظمة و الدقيقة . وأعطينا صفات متنوعـــة للجبـــر المضـــاعف $M(A)$،كالتالي :

إذا كانت A جبر إبدالي لها عنصر محايد ، فإن $M(A)$متماثلـــة جبريـــاً مـــع الجبـــر المكون من الممثلات المنتظمة اليسرى $\{L_x : x \in A\}$. إذا كانت A جبر باناخ الإبدالي (أو بشكل أعم جبر فيرشيت) فإن كل $T \in M(A)$ محددة بدالة مركبة متصـــلة علـــى الفضاء المثالي الأكبر $\Delta(A)$ من A والتي تكون محدودة إذا كانت A جبر باناخ شـــبة البسيط . بشكل خاص ، إذا كانت $A = C_0(X)$ مـــع فضـــاء هاوســـدورف المتـــراص موضعياً X، فإن $M(A) = C_b(X)$. بالإضافة إلـــى ذلـــك ، فـــإن التطبيـــق الخطـــي $T : A \to A$ يكون مضاعِف $T \Leftrightarrow$ تكون "مضاعِف جوردان ". ودرسنا أيضاً خواصـــاً متنوعة للفضاء المثالي الأكبر $\Delta(M(A))$ مـــن الجبـــر المضـــاعِف $M(A)$ بالنســـبة لتبولوجيات جلفاند وغطاء النواة . ليس بالضرورة أن تكـــون مجموعـــة المضـــاعفات المتماثلة فضاءاً متجهاً ; وإنما هي تؤلـــف زمـــرة هاوســـدورف التبولوجيـــة بالنســـبة للتبولوجي الدقيق وأيضاً تكون مجموعة المضاعفات المتماثلة متراصة في ظل بعـــض الفروض الإضافية .عندما تكون A جبراً تبولوجياً مع أساس شاودر العمـــودي ، فـــإن $M(A)$ ممكن أن تدمج اتصالياً داخل الجبر s من متتابعة الأرقام المركبـــة; بالإضـــافة إلى ذلك ، إذا كان لـــ A عنصراً محايـــداً وأساســـاً عموديـــاً غيـــر شـــرطي ، فـــإن $M(A) \cong l_\infty$. تشمل النتائج السابقة، بالإضافة إلى جهد الآخرين، جهد كلٍ مـــن وانـــج في عام ١٩٦١ م وبيرتل في عامي ١٩٦١ م و١٩٦٢ م و جونسون في عامي ١٩٦٤م و ١٩٦٦م و ماتي في عام ١٩٦٥م وبوزبي في عام ١٩٦٨م. و في حالة جبور بانـــاخ عمل لارسن في عام ١٩٧١م وكذلك مجهودات حسين الحديثة في عام ١٩٨٩م ، وأيضاً عمل كلّ من خان ومحمد وثاهيم عام ١٩٩٩م في الحالة العامة لجبور فيرشيت.

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my parents, my husband, and
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MULTIPLIERS ON COMMUTATIVE FRECHET ALGEBRAS

CONTENTS

CHAPTER 4 : MULTIPLIERS ON COMMUTATIVE FRÉCHET ALGEBRAS

INTRODUCTION

The study of multipliers is a relatively old area of mathematical research in which there has been a considerable activity in recent past. The concept of a *multiplier* (also called a *centralizer*) first appears in Harmonic Analysis in connection with the theory of summability for Fourier series. Subsequently the notion has been employed in other areas of Harmonic Analysis, as in the investigation of homomorphisms of group algebras, in the general theory of Banach algebras, in the theory of semigroup of operators and in the study of partial differential equations.

We shall be mainly concerned with the general notion of multipliers for commutative Banach and topological algebras. This was first studied by Helgason (1956) followed by Wang (1961), Birtel (1961, 1962), Johnson (1964), Busby (1968), Larsen (1971), Husain (1989), among others (see the list of references). The present characterization of the *multiplier algebras* is due to Wang (1961). He showed that if A is a commutative semisimple Banach algebra, then each $T \in M(A)$ is determined by a continuous bounded complex-valued function on the *maximal ideal space* $\Delta(A)$ of A. In particular, if $A = C_0(X)$ with X a locally compact Hausdorff space, then $M(A) = C_b(X)$. A characterization of a "multiplier" has been given by Birtel (1961). He showed that a linear mapping $T : A \to A$ is a multiplier $\Leftrightarrow$ it is a *Jordan multiplier.*

Johnson (1964) introduced the notions of *left multipliers, right multipliers*, and *multipliers* both on algebras and topological algebras. If A is a faithful commutative algebra, then $M_l(A) = M_r(A) = M(A)$; if, in addition, A has an identity, then $M(A)$ is algebraically isomorphic with the algebra $\{L_x : x \in A\}$ of regular representations. Further, an algebra A can be embedded as (left, right) ideal into the multiplier algebras $M_l(A), M_r(A)$ and $M(A)$. If A is a Banach algebra, then, under certain conditions (such as A is faithful or has an approximate identity), Wang (1961) and Johnson (1966) established the linearity and continuity of the multipliers $T : A \to A$. Wang (1961) and Busby (1968) also introduced the *uniform* and *strict* topologies on the multiplier algebras and studied their completeness and dense properties.

The *maximal ideal space* of the multiplier algebra was studied by Foias (1958) and subse-

quently by Wang (1961), Máté (1965) and Husain (1989). Here, we study various properties of the maximal ideal space $\Delta(M(A))$ of the multiplier algebras $M(A)$ and the relation between $\Delta(A)$ and $\Delta(M(A))$ with respect to both the Gelfand topology w^* and hull-kernel topology τ_{hk}.

The notion of *isometric multipliers* appears first in the papers of Wermer (1953) and Birtel (1962). We see that the set of isometric multipliers need not be a vector space; however, it forms a Hausdorff topological group with respect to the strict topology and is compact under some additional assumptions.Several authors have also studied multipliers by considering the multiplier algebra $M(A)$ as a commutative extension of a commutative Banach algebra A.

We have also studied multipliers in the more general setting of commutative Fréchet algebras. In particular, we present some extensions of above results from Banach algebras to topological algebras, due to Husain (1989) and Khan-Mohammed-Thaheem (1999). In particular, if A is a topological algebra with an *orthogonal Schauder basis*, $M(A)$ can be continuously embedded into the algebra s of sequences of complex numbers; if, in addition, A has an identity and an *unconditional* orthogonal basis, then $M(A) \cong l_\infty$.

Chapter 1 consists of basic definitions and results from *Topology, Topological Vector Spaces, and Banach and Topological Algebras.* The proofs of results in Sections 1.1-1.4 are omitted since they are available in any standard book on these topics. However, we have included proofs of some results in Sections 1.5 and 1.6 as these are specialized topics.

In Chapter 2, we study the notions of left multipliers, right multipliers, and multipliers on both algebras and Banach algebras. In Section 2.1, we give basic definitions and properties of various types of multipliers on algebras. Section 2.2 deals with the problem of embedding of an algebra into the algebras of multipliers as (left, right) ideal. In Section 2.3, we establish the linearity and continuity of the multipliers using strong results like the closed graph theorem and the Cohen's factorization theorem.We also introduce the uniform and strict topologies on the multiplier algebras and study their completeness and dense properties.

Chapter 3 deals with the commutative theory of multipliers on Banach algebras. In Section 3.1, we give a characterization of the multiplier algebra as a cetain function space of continuous bounded complex-valued functions on a locally compact Hausdorff space. In Section 3.2, we study various properties of the maximal ideal space of multiplier algebras. We consider

isometric multipliers in Section 3.3 and discuss their properties.

In Chapter 4, we study multipliers in the more general setting of commutative Fréchet algebras. In particular, we present extensions of some results of Chapers 2 and 3 from Banach algebras to topological algebras in Sections 4.1-4.2. In Section 4.3, we obtain further results when the algebras have orthogonal Schauder bases and unconditional bases.

CHAPTER I

BASIC THEORY OF TOPOLOGICAL VECTOR SPACES AND TOPOLOGICAL ALGEBRAS

This chapter consists of basic definitions and results from *Topology, Topological Vector Spaces, Banach Algebras and Topological Algebras.* The proofs of results in Sections 1.1-1.4 are omitted since they are available in any standard book on these topics (see the list of references). However, we have included proofs of some results in Sections 1.5 and 1.6 as these are specialized topics.

1.1 Topological Spaces

In this section, we present some definitions and results in topological spaces; the details may be found in the books of Kelley [17] and Simmons [25].

Definition 1.1.1. Let X be a given set and τ a collection of subsets of X. Then τ is called a *topology* on X if it satisfies the following conditions:

(a) $X,\phi \in \tau$.

(b) The union of an arbitrary number of sets in τ is also in τ.

(c) The intersection of a finite number of sets in τ is also in τ.Then X with the topology τ is called a *topological space* and is denoted by (X,τ). The members U of τ are called *τ-open sets* (or simply *open sets*) in X.

Definition 1.1.2. Let σ and τ be two topologies on a set X. If $\sigma \subseteq \tau$, then σ is said to be *weaker* (or *coarser*) than τ and we write $\sigma \leq \tau$; in this case τ is called *stronger* (or *finer*) than σ.

Definition 1.1.3. Let X be any non-empty set.

(i) If $\tau = P(X)$, the collection of all subsets of X, Then τ is a topology on X, called the *discrete topology* on X. Clearly, this is the largest of all topologies on X.

(ii) If $\tau = \{X, \phi\}$, then τ is a topology on X, called the *indiscrete topology* on X. Clearly, this is the smallest of all topologies on X.

Definition 1.1.4. Let A be a non-empty subset of a topological space (X, τ), and let

$$\tau_A = \{U \cap A : U \in \tau\}.$$

Then τ_A is a topology on A, called the *relative* (or, *induced*) *topology* on A.

Definition 1.1.5. Let (X, τ) be a topological space.

(a) A subset β of τ is said to be a *basis* of the topology τ if each τ-open set is the union of sets in β.

(b) A subset γ of τ is called a *sub-basis* of τ if finite intersections of sets in γ form a basis of the topology τ.

Definition 1.1.6. Let (X, τ) be a topological space and $x \in X$. A set $U \subseteq X$ is said to be a *neighbourhood* (or a *τ-neighbourhood*) of x if there exists a set $V \in \tau$ such that $x \in V \subseteq U$. The collection $\mathcal{N}_x$ of all τ-neighbourhoods of x is called the *system of τ- neighbourhoods* of x.

Clearly a non-empty open set V is a neighbourhood of each of its elements. Further, a set V is open if and only if it contains a neighbourhood of each of its elements.

Theorem 1.1.7. *Let (X, τ) be a topological space. For any $x \in X$, let $\mathcal{N}_x$ be the system of τ-neighbourhoods of x. Then:*

(NB_1) $\mathcal{N}_x$ is non-empty and $x \in U$ for all $U \in \mathcal{N}_x$.

(NB_2) If $U \in \mathcal{N}_x$ and $U \subseteq W$, then $W \in \mathcal{N}_x$.

(NB_3) Each finite intersection of sets in $\mathcal{N}_x$ belongs to $\mathcal{N}_x$

(NB_4) If $U \in \mathcal{N}_x$, then there is a $V \in \mathcal{N}_x$ such that $V \subseteq U$ and $U \in \mathcal{N}_y$ for each $y \in V$.

Conversely, given a set X and that for each $x \in X$ is given a non-empty collection $\mathcal{N}_x$ of subsets of X satisfying the above conditions (NB_1) (NB_2), then one can define a unique topology τ on X such that $\mathcal{N}_x$ is the system of τ-neighbourhoods of x for each $x \in X$.

Definition 1.1.8. Let $\mathcal{N}_x$ be the system of neighbourhoods of $x \in (X, \tau)$. A subfamily $\mathcal{B}_x$ of $\mathcal{N}_x$ is said to be a *local basis (or local base) at x* if, for each $U \in \mathcal{N}_x$, there exists a $V \in \mathcal{B}_x$ such that $V \subseteq U$.

Definition 1.1.9. Let X be a non-empty set. Then a function $d : X \times X \to \mathbb{R}$ is called a *metric* (or a *distance function*) on X if, for every $x, y, z \in X$,

(M_1) $d(x,y) \geq 0$;

(M_2) $d(x,y) = 0 \Leftrightarrow x = y$;

(M_3) $d(x,y) = d(y,x)$;

(M_4) $d(x,z) \leq d(x,y) + d(y,z)$.

In this case, the pair (X,d) is called a *metric space.*

Definition 1.1.10. Let (X,d) be a metric space. For any $x \in X$ and $r > 0$, the set

$$B_r(x) = \{y \in X : d(y,x) < r\}$$

is called an *open ball* of radius r and center x. A subset A of X is said to be *open* in (X,d) (or *d-open*) if, for each $x \in A$, there exists an $r > 0$ such that $B_r(x) \subseteq A$. The collection t_d of all d-open sets in X forms a topology on X, called the *metric topology induced by the metric* d. In other words, each metric space (X,d) is a topological space with the topology t_d. In this case, for any $x \in X$, the collection

$$\mathcal{B}_x = \{B_r(x) : r > 0 \text{ and } r \text{ is rational }\}$$

of open balls gives a countable local basis at x.

Definition 1.1.11. A topological space (X,τ) is said to be *metrizable* if there exists a metric d on X such that $\tau = t_d$.

Definition 1.1.12. (a) A sequence $\{x_n : n \in \mathbb{N}\}$ in a metric space (X,d) is called a *Cauchy sequence* if, given any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$d(x_n, x_m) < \varepsilon \qquad \text{for all } n, m \geq N.$$

(b) A sequence $\{x_n : n \in \mathbb{N}\}$ in a metric space $(X,d)$is said to be *convergent* to $x \in X$, if given any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$d(x_n, x) < \varepsilon \qquad \text{for all } n \geq N.$$

In this case, we denote $x_n \to x$. Clearly, every convergent sequence is a Cauchy sequence. But the converse need not be true.

Definition 1.1.13. Let X be a topological space. A net $\{x\alpha : \alpha \in I\} \subseteq X$ is said to be *convergent* to $x \in X$, written as $x\alpha \longrightarrow x$, if, for any neighbourhood U of x in X, there exists an index $\alpha_o \in I$ such that $x\alpha \in U$ for all $\alpha \geq \alpha_o$.

Definition 1.1.14. A set A in (X, τ) is said to be τ-*closed* (or simply *closed*) if $X \setminus A$, the complement of A in X, is τ-open.

Theorem 1.1.15. *Let (X, τ) be a topological space. Then:*

(a) X and ϕ are τ-closed.

(b) The union of any finite number of τ-closed sets is τ-closed.

(c) The intersection of an arbitrary number of τ-closed sets is τ-closed.

Definition 1.1.16. Let (X, τ) be a topological space and $A \subseteq X$. An element $x \in A$ is called an *interior point* of A if there exists an open set U such that $x \in U \subseteq A$. The set of all interior points of A is called the *interior* of A and is denoted by A^0 (or $int(A)$). Clearly $A^0 \subseteq A$.

Theorem 1.1.17. *Let (X, τ) be a topological space and $A \subseteq X$. Then:*

(a) A^0 is an open subset of X.

(b) A is open $\Leftrightarrow A = A^0$.

(c) $A^0 =$ union of all open sets contained in A.

Definition 1.1.18. Let (X, τ) be a topological space and $A \subseteq X$. An element $x \in X$ is called a *limit point* (or an *accumulation point*) of A if every open set containing x contains a point of A other than x. The set of all limit points of A is called the *derived set* of A and is denoted by A_d. The set $A \cup A_d$ is called the *closure* of A and is denoted by $\overline{A}$ (or $cl\ (A)$). Clearly $A \subseteq \overline{A}$.

Theorem 1.1.19. *Let (X, τ) be a topological space and $A \subseteq X$. Then:*

(a) $\overline{A}$ is a closed subset of X.

(b) A is closed $\Leftrightarrow A = \overline{A}$.

(c) $\overline{A} =$ intersection of all closed sets which contain A.

(d) *For any $x \in X$, $x \in \overline{A} \Leftrightarrow$ there exists a net $\{x\alpha\} \subseteq A$ such that $x\alpha \longrightarrow x$.*

Definition 1.1.20. Let (X, τ) be a topological space and $A \subseteq X$. Then the set $\partial A = \overline{A} \setminus A^0$ is called the *boundary* of A; equivalently, $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Definition 1.1.21. Let (X, τ) be a topological space and $A \subseteq X$. If $\overline{A} = X$, then we say that A is τ-*dense* (or simply, *dense*) in X.

Definition 1.1.22. A topological space (X, τ) is called

(a) *Hausdorff* if, for any two points $x \neq y$ in X, there exist two open sets $U, V \subseteq X$ such that $x \in U, y \in V$, and $U \cap V = \phi$;

(b) *normal* if, for any two closed sets $A, B \subseteq X$ with $A \cap B = \phi$, there exist two open sets $U, V \subseteq X$ such that $A \subseteq U, B \subseteq V$, and $U \cap V = \phi$;

(c) *completely regular* if, for any closed set $A \subseteq X$ and $x \notin A$, there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$ (i.e. $f(y) = 1$ for all $y \in A$);

(d) *first countable* if, for each $x \in X$, there exists a countable local base at x;

(e) *second countable* if there exists a countable base for τ;

(f) *separable* if it has a countable dense subset;

(g) *complete* if it is a metric space and every Cauchy sequence in X is convergent to a point in X.

Definition 1.1.23. Let (X, τ) be a topological space and $A \subseteq X$. A collection $\mathcal{U} = \{U\alpha : \alpha \in I\}$ of open subsets of X is called an *open cover* of A if

$$A \subseteq \cup_{\alpha \in I} U\alpha;$$

if there exists a finite subset $\mathcal{V} = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$ of $\mathcal{U}$ such that $A \subseteq \cup_{i=1}^{n} U_{\alpha_i}$, then $\mathcal{V}$ is called a *finite subcover* of A..

Definition 1.1.24. A topological space X is called

(a) *compact* if every open cover of X has a finite subcover;

(b) *countably compact* if every countable open cover of X has a finite subcover;

(c) *a Lindelöf space* if every open cover of X has a countable subcover;

(d) *sequentially compact* if every sequence in X has a convergent subsequence;

(e) *locally compact* if each $x \in X$ has a relatively compact neighbourhood;

(f) *σ-compact* if it can be written as a countable union of compact sets;

(g) *k-space* if, for any $U \subseteq X$, U is open in X whenever $U \cap K$ is open in K for each compact $K \subseteq X$.

Every Hausdorff locally compact (in particular, compact) space is normal, completely regular and a k-space. All metric spaces are k-spaces and all compact metric spaces are complete. A metric space X is compact $\Leftrightarrow$ it is countably compact $\Leftrightarrow$ it is sequentially compact. In a metric space, the separable, second countable, and Lindelöf properties are equivalent. If X is a locally compact and σ-compact Hausdorff topological space, then it can be expressed as $X = \bigcup_{n=1}^{\infty} K_n$, where each $K_n \subseteq int\ K_{n+1}$.

Definition 1.1.25. Let $\{(X\alpha, \tau\alpha) : \alpha\varepsilon I\}$ be an indexed family of topological spaces. The *product topology* on the cartesian product $\prod_{\alpha\varepsilon I} X\alpha$ is defined as the topology τ obtained by taking as its base the collection of all sets of the form $\prod_{\alpha\varepsilon I} U\alpha$, where each $U\alpha$ is $\tau\alpha$-open in $X\alpha$ and $U\alpha = X\alpha$ except for a finite number of indices α.

Theorem 1.1.26. (Tychonoff). *If $\{X\alpha : \alpha\varepsilon I\}$ is a family of compact spaces, then $\prod_{\alpha\varepsilon I} X\alpha$, with the product topology, is a compact space.*

Definition 1.1.27. Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is said to be *continuous* at $x_o \in X$ if, for each neighbourhood V of $f(x_o)$ in Y, there exists a neighbourhood U of x_o in X such that $f(U) \subseteq V$; f is said to be *continuous* on X if it is continuous at each $x \in X$.

The following theorem gives some equivalent conditions for continuity.

Theorem 1.1.28. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$. Then the following conditions are equivalent:*

(a) *f is continuous on X.*

(b) *For each open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.*

(c) *For each closed set $W \subseteq Y$, $f^{-1}(W)$ is closed in X.*

(d) $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

(e) *For any $x \in X$ and any net $\{x\alpha\} \subseteq X$ with $x\alpha \longrightarrow x$ in X, $f(x\alpha) \longrightarrow f(x)$ in Y.*

Definition 1.1.29. Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is said to be a *homeomorphism* if

(a) f is continuous,

(b) f is one-one and onto,

(b) $f^{-1} : Y \to X$ is continuous.In this case, X and Y are said to be *homeomorphic.*

Definition 1.1.30. Let X and Y be topological spaces. Then;

(i) A function $f : X \to Y$ is said to be an *open function* on X, if for each open set U in X, $f(U)$ is an open set in Y.

(ii) A function $g : X \to Y$ is called a *closed function* if for each closed set U in X, $g(U)$ is a closed set in Y.

In general, a continuous function $\not\Leftrightarrow$ an open function $\not\Leftrightarrow$ a closed function. However, the following holds.

Theorem 1.1.31. *Let X and Y be topological spaces and $f : X \longrightarrow Y$ a one-one and onto function. Then the following conditions are equivalent:*

(a) f is continuous and an open function.

(b) f is continuous and a closed function.

(c) f is a homeomorphism.

Definition 1.1.32. Let (X, d) and (Y, d') be metric spaces. Then a function $f : X \to Y$ is said to be an *isometry* of X into Y if

$$d'(f(x), f(y)) = d(x, y) \qquad \text{for all } x, y \in X;$$

that is, f preserves distance between points from X into Y. The metric spaces (X, d) and (Y, d') are said to be *isometric* if there exists a one-one isometry f of X onto Y. In this case, f is clearly a homeomorphism.

The following three results are fundamental regarding the existence and extension of continuous functions on topological spaces.

Theorem 1.1.33. *(Urysohn) Let X be a normal space, and let A and B be two disjoint closed subsets of X. Then there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.*

Theorem 1.1.34. (Tietze) *Let X be a normal space, and let A be a closed subset of X. Then, for any continuous function $f : A \longrightarrow [0,1]$ (or $\mathbb{R}$), there exists a continuous function $g : A \longrightarrow [0,1]$ (or $\mathbb{R}$) such that $g = f$ on A.*

(Here g is called a *continuous extension* of f from A to X).

Definition 1.1.35. Let X be a topological space. A function $f : X \longrightarrow \mathbb{K}$ ($\mathbb{R}$or $\mathbb{C}$) is said to *vanish at infinity* if, for each $\varepsilon > 0$, the set $F\varepsilon = \{x \in X :| f(x) | \geq \varepsilon\}$ is "relatively compact" in X (i.e. $F\varepsilon$ has compact closure in X). The *support* of $f : X \longrightarrow \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is defined as the closure of the set $\{x \in X : f(x) \neq 0\}$ in X.

Definition 1.1.36. For any topological space X, let $C(X) = C(X, \mathbb{R})$ be the set of all continuous $\mathbb{K}$-valued functions on X and $C_b(X)$ (resp. $C_o(X)$, $C_{oo}(X)$) the subset of $C(X)$ consisting of those functions which are bounded (resp. vanish at infinity, have compact support). Clearly, $C_{00}(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X)$ with $C_{00}(X) = C(X) \Leftrightarrow X$ is compact. All of these are vector spaces over the field $\mathbb{K}$. Further, each of the space $\mathcal{C} = C_{00}(X), C_0(X)$, or $C_b(X)$ is a metric space (hence a topological space) with respect to the metric d given by

$$d(f,g) = \sup\{| f(x) - g(x) |: x \in X\}, \qquad f, g \in \mathcal{C}.$$

(Equivalently, $\mathcal{C}$ is a normed vector space with norm $\|.\|$defined by

$$\|f\| = d(f,0) = \sup\{| f(x) |: x \in X\}, \qquad f \in \mathcal{C}.)$$

The topology induced by metric d on $\mathcal{C}$ is called the *uniform topology* and is denoted by u. Note that the above metric d is not well-defined on $C(X)$ since $d(f,g)$ may not exist for unbounded $f, g \in C(X)$. However, $C(X)$ is a metric space with respect to the metric d^* defined as

$$d^*(f,g) = \min\{1, d(f,g)\}, \qquad f, g \in C(X).$$

Definition 1.1.37. Let (X, τ) be a non-compact topological space.A compact space (X^*, τ^*) is called a *compactification* of X if there exists a function $e : X \to X^*$ such that

(a) $e : X \to e(X)$ is a homeomorphism;

(b) $e(X)$ is τ^*-dense in X^*.

We now describe two well known compactifications.

Definition 1.1.38. Let (X, τ) be a non-compact topological space, and let $X_\infty = X \cup \{\infty\}$, where ∞ is an element or object not in X. (We may take $\infty = X$ since $X \notin X$.) Define

$$\tau_\infty = \tau \cup \{V \subseteq X_\infty : \infty \in V \text{ and } X_\infty \setminus V \text{ is closed and compact in } X\}.$$

Then τ_∞ is a topology on X_∞ and (X_∞, τ_∞) is compact, called the *one-point compactification* (or *Alexandroff compactification*) of (X, τ). Clearly, $\tau \subseteq \tau_\infty$ and $\tau_\infty = \tau$ on X. Further, (X_∞, τ_∞) is Hausdorff $\Leftrightarrow$ (X, τ) is Hausdorff and locally compact. As an example, if $X = (0, 1)$, then $X_\infty = S^1$, the unit circle in $\mathbb{R}^2$ with centre at the origin and taking ∞ as the point $(1, 0)$ in $\mathbb{R}^2$.

Definition 1.1.39. Let (X, τ) be a completely regular Hausdorff space, and let $C = C_b(X, \mathbb{R})$, the set of all continuous and bounded functions $f : X \to \mathbb{R}$. For each $f \in C$, define a closed bounded (hence compact) interval in $\mathbb{R}$ by

$$I_f = [\inf\text{-}f(X),\ \sup\text{-}f(X)].$$

Define a map $e : X \to \prod_{f \in C} I_f$ (with the product topology) by

$$e(x) = (\, f(x)\,)_{f \in C}, \qquad x \in X.$$

Then $\beta X = \overline{e(x)}$ is a compact space, called the *Stone-Cech compactification* of X.

Theorem 1.1.40. (Stone-Cech) *Let X be a completely regular Hausdorff space and βX its Stone-Cech compactification. Then:*

(I) *X is dense in βX;*

(ii) Every function f in $C_b(X)$ has a unique norm-preserving extension to a function $\widehat{f}$ in $C(\beta X)$ (i.e. .$\|f\| = \|\widehat{f}\|$).

(iii) For any compact Hausdorff space Y, each continuous function $f : X \longrightarrow Y$ has a unique continuous extension $\widehat{f} : \beta X \longrightarrow Y$.

Theorem 1.1.41. *Let X be a locally compact Hausdorff space, K a compact subset of X, and U a neighbourhood of K. Then there exists a continuous function $\phi : X \longrightarrow [0,1]$ such that $\phi(K) = 1$ and the support of ϕ is compact and contained in U. (The support of ϕ is the closure of the set $\{x \in X : \phi(x) = 0\}$).*

Definition 1.1.42. Let X be a topological space. A subset A of $C(X)$ is said to *separate the points of X* if, for any $x \neq y$ in X, there exists an $f \in A$ such that $f(x) \neq f(y)$; we say that A *does not vanish on X* if, for each $x \in X$, there exists some $g \in A$ such that $g(x) \neq 0$; A is said to be *self-adjoint* if, for each $g \in A$, its complex conjugate $\overline{g}$ also belongs to A; A is called an *ideal* in $C(X)$ if $fg \in A$ for all $f \in A$ and $g \in C(X)$.

The classical Weierstrass approximation theorem, proved in 1885 by K. Weierstrass, states that the set of all polynomials in $t \in [0,1]$ with real coefficients is uniformly dense in $C[0,1] = C([0,1], \mathbb{R})$. In 1948, M. H. Stone obtained the following far reaching generalization of this theorem.

Theorem 1.1.43. (Stone-Weierstrass). *Let X be a locally compact Hausdorff space. Suppose a subset A of $C_0(X)$ satisfies the following conditions:*

(i) A is a self-adjoint subalgebra of $C_0(X)$;

(ii) A separates the points of X;

(iii) A does not vanish on X.

Then A is u-dense in $C_0(X)$.

This theorem has important applications in several branches of mathematics. In particular, it plays a crucial roll in the proof of Gelfand-Naimark theorem in the theory of Banach algebras.

1.2. Normed Spaces and Topological Vector Spaces

We first give some definitions and results from normed spaces and topological vector spaces (TVS); for detail, see the book of Rudin [23] or of Schaefer [24]. In the sequel, we consider all vector spaces and algebras over the field $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$.

Definition 1.2.1. A set E with two operations of addition $(x, y) \to x + y$ and scalar multiplication $(\lambda, x) \to \lambda x$. is called a *vector space* over the field $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$ if the following conditions hold:

(a) E is an abelian group under addition,

(b) For any $x, y \in E$ and $\lambda, \mu \in \mathbb{K}$, the scalar multiplication satisfies:

(i) $\lambda(x + y) = \lambda x + \lambda y$

(iii) $(\lambda + \mu)x = \lambda x + \mu x$,

(ii) $(\lambda\mu)x = \lambda(\mu x)$,

(iv) $1.x = x$, where 1 is the multiplicative identity of $\mathbb{K}$.

Definition 1.2.2. Let E be a vector space over $\mathbb{K}$. Then a function $\|.\| : E \to \mathbb{R}$ is called a *norm* on E if

(N_1) $\|x\| \geq 0$ for all $x \in E$;

(N_2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N_3) $\|\lambda x\| = |\lambda|\,\|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$ (*absolutely homogeneous*);

(N_4) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$ (*subadditive*).

(We mention that (N_1) and (N_2) actually follow from (N_3) and (N_4), but we state them here for clarity and later reference).

A vector space E with a norm $\|.\|$ is called a *normed vector space* or, simply, a *normed space.* Clearly, every normed space $(E, \|.\|)$ is a metric space with metric given by

$$d(x, y) = \|x - y\|, \quad x, y \in E.$$

A normed space E is called a *Banach space if it is* complete with respect to the metric induced by the norm (i.e. if every Cauchy sequence in E converges to an element in E)

Definition 1.2.3. Let E be a vector space over $\mathbb{K}$. A subset A of E is said to be

(i) *absorbing* if, for each $x \in E$, there exists a number $r = r(x) > 0$ such that $x \in \lambda\ A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \geq r$,

(ii) *balanced* if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$,

(iii) *convex* if $\lambda x + (1-\lambda)y \in A$ for all $x, y \in A$ and $0 < \lambda < 1$.

Definition 1.2.4. Let E be a vector space over $\mathbb{K}$ and let τ be a topology on E. Then (E, τ) is called a *topological vector space* (in short, TVS) if the following conditions hold:

(TVS1) The operation of addition $(x, y) \to x + y$ of $E \times E \to E$ is jointly continuous; i.e., given any $x, y \in E$ and any τ neighbourhood U of $x + y$ in E, there exist τ neighbourhoods V of x and W of y in E such that $V + W \subseteq U$.

(TVS2) The operation of scalar multiplication $(\lambda, x) \to \lambda x$ of $\mathbb{K} \times E \to E$ is jointly continuous; i.e., .given any $x \in E$ and $\lambda \in \mathbb{K}$ and any τ neighbourhood U of λx in E, there exist a τ neighbourhood V of x in E and a neighbourhood D of λ in $\mathbb{K}$ such that $DV \subseteq U$.

Example: Any normed space $(E, \|.\|)$ is a TVS since the conditions (TVS1) and (TVS2) follow, respectively, from:

(i) If $x, y \in E$ and $x_n \to x$, $y_n \to y$, then $x_n y_n \to xy$ in E.

(ii) If $x \in E$, $\lambda \in \mathbb{K}$ and $\lambda_n \to \lambda$, $x_n \to x$, then $\lambda_n x_n \to \lambda x$ in E.

Theorem 1.2.5. *Let (E, τ) be a TVS. Then:*

(a) (E, τ) has a base $\mathbb{W}$ of neighbourhoods of 0 consisting of closed, balanced, and absorbing sets.

(b) (E, τ) is Hausdorff if and only if $\bigcap_{G \in \mathbb{W}} G = \{0\}$.

Definition 1.2.6. A TVS (E, τ) is called a *locally convex space* (in short, LCS) if it has a base of convex neighbourhoods of 0 in E.

Definition 1.2.7. Let (E, τ) be a TVS. Then

(a) A net $\{x\alpha : \alpha \in I\}$ in E is said to be *convergent* to $x \in E$ if, given any neighbourhood U of 0 in E, ther exists an $\alpha_0 \in I$ such that $x\alpha - x \in U$ for all $\alpha \geq \alpha_0$. In this case, we write $x\alpha \to x$.

(b) A net $\{x\alpha : \alpha \in I\}$ in E is said to be a *Cauchy net* if, given any neighbourhood U of 0 in E, ther exists an $\alpha_0 \in I$ such that $x\alpha - x\beta \in U$ for all $\alpha, \beta \geq \alpha_0$.

(c) (E, τ) is said to be *complete* if every Cauchy net in E is convergent to a point in E.

Definition 1.2.8. Let E be a vector space over $\mathbb{K}$ and $q : E \to \mathbb{R}$ a function. Then

(1) q is called a *seminorm* on E if it satisfies

(a) $q(x) \geq 0$ for all $x \in E$;

(b) $q(x) = 0$ if $x = 0$;

(c) $q(\lambda x) = |\lambda|\, q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ (*absolutely homogeneous*);

(d) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$ (*subadditive*).

(We mention that (a) and (b) actually follow from (c) and (d), but we state them here for clarity and later reference).

(2) q is called an *F-seminorm* on E if it satisfies

(a) $q(x) \geq 0$ for all $x \in E$;

(b) $q(\lambda x) \leq q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(c) $q(\lambda x) \to 0$ for all $x \in \mathbb{K}$ with $\lambda \to 0$;

(d) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$ (*subadditive*).

(3) q is called an *F-norm* on E if it satisfies

(a) $q(x) \geq 0$ for all $x \in E$;

(b) $q(x) = 0 \Leftrightarrow x = 0$;

(c) $q(\lambda x) \leq q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(d) $q(\lambda x) \to 0$ for all $x \in \mathbb{K}$ with $\lambda \to 0$;

(e) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$ (*subadditive*)

(4) q is called a *p-norm* on E, where $0 < p \leq 1$, if it satisfies

(a) $q(x) \geq 0$ for all $x \in E$;

(b) $q(x) = 0 \Leftrightarrow x = 0$;

(c) $q(\lambda x) = |\lambda|^p\, q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$;

(*d*) $q(x+y) \leq q(x) + q(y)$ for all $x, y \in E$ (*subadditive*).

Note: If $q : E \to \mathbb{R}$ satisfies $q(\lambda x) = \lambda q(x)$ for all $\lambda \geq 0$, then q is called *positively homogeneous*. It may happen that $q(x) = 0$ for some $x \neq 0$. If a seminorm (*F-seminorm*) q satisfies $q(x) = 0$ implies $x = 0$ (or equivelently $x \neq 0$ implies $q(x) \neq 0$), then q is a *norm (F-norm)* on E.

Lemma 1.2.9. *Let q be a seminorm on a vector space E. Then, for each $r > 0$, $V_r = \{x \in E : q(x) \leq r\}$ is an absorbing, balanced and convex subset of E.*

Definition 1.2.10. If A is any absorbing subset of a vector space E, then the function $\rho_A : E \to \mathbb{R}$ defined by

$$\rho_A(x) = \inf\{t > 0, \ \ x \in tA\}, \ \ x \in E,$$

is called the *Minkowski functional* of A.

Lemma 1.2.11. *Let A be an absorbing, balanced and convex subset of a vector space E. Then its Minkowski functional ρ_A is a seminorm on E.*

Theorem 1.2.12. If Q is a family of seminorms on a vector space E over $\mathbb{K}$, then Q defines a locally convex topology, say, τ on E and a base of τ-neighbourhoods of 0 consists of all sets of the form $\{x \in E : \max_{1 \leq j \leq n} p_j(x) \leq \varepsilon\}$, where $\varepsilon > 0$ and $p_j \in Q$, $n \in \mathbb{N}$. Conversely, if (E, τ) is an LCS, then the topology τ can be defined by a family of seminorms, namely, the Minkowski functionals of balanced convex τ-neighbourhoods of 0.

Definition 1.2.13. Let (E, τ) be a TVS. The topological dual E' of E is defined as the set of all continuous linear functionals on E. Suppose E' separates points of E (i.e. if $x \in E$ with $x \neq 0$, there exists an $f \in E'$ such that $f(x) \neq 0$). For any $f \in E', p_f(x) = |f(x)|, x \in E$, defines a seminorm p_f on E. Hence $\{p_f : f \in E'\}$ is a family of seminorms on E and, so by Theorem 1.2 12, this defines a locally convex topology on E. This topology is called *weak topology* on E and is denoted by $w = \sigma(E, E')$. We can similarly define the *weak**- *topology*, denoted by $w^* = \sigma(E',)E$, which is generated by the family $\{q_x : x \in E\}$ of seminorms on E', where $q_x(f) = |f(x)|$ for all $f \in E'$.

Theorem 1.2.14. *A TVS* (E,τ) *is metrizable if and only if it is Hausdorff and has a countable base of* τ*-neighbourhoods of* 0. *In this case* τ *may be defined by a metric* d *such that*

(i) $d(x+z,\ y+z) = d(x,y) = d(x-y,0)$ *for all* $x,y,z \in E$ (i.e. *d is translation invarient)* ,

(ii) $d(\lambda x,0) \leq d(x,0)$ *for all* $x \in E$ *and* $|\lambda| \leq 1$, *and so* τ *is given by an* F*-norm* q, *where* $q(x) = d(x,0)$ *for all* $x \in E$.

Definition 1.2.15. A complete metrizable TVS is called an *F-space.*

Clearly, every Banach space is an F-space

Definition 1.2.16. A subset A of a TVS E is called

(a) *bounded* if, for any neighbourhood U of 0 in E, there exists $r > 0$ such that $A \subseteq \lambda U$ for all $|\lambda| \leq r$.

(b) *totally bounded (or precompact)* if for each neighbourhood U of 0 there exists a finite subset N of E such that $A \subseteq N + U$.

Definition 1.2.17. A TVS E is said to be *locally bounded* if E possesses a bounded neighbourhood of 0. If V is a bounded neighbourhood of E, then clearly, $\{\frac{1}{n}V : n \geq 1\}$ is a countable base of neighbourhoods of 0 in E.

Theorem 1.2.18. *(a) Every Hausdorff locally bounded TVS is metrizable.*

(b) The topology of a Hausdorff locally bounded TVS can always be generated by a quasi-norm (or equivalently by a p-norm from some p, $0 < p \leq 1$).

Theorem 1.2.19. *Let* E *and* F *be TVSs and* $T : E \to F$ *a linear mapping. If* T *is continuous, then* T *maps bounded sets into bounded sets; the converse holds if* E *is metrizable.*

Definition 1.2.20. Let E and F be TVSs, and let $CL(E,F)$ denote the vector sace of all continuous linear mappings from $E \to F$. A subset H of $CL(E,F)$ is said to be *equicontinuous* if for each neighbourhood V of 0 in F, there exists a neighbourhood U of 0 in E such that $T(U) \subseteq V$ for all $T \in H$.

Theorem 1.2.21. (Principle of uniform boundedness) *Let* E *be a complete metrizable TVS and* F *any TVS. Let* $A = \{T\alpha : \alpha \in I\}$ *be a collection of continuous linear mappings from* E *into* F *such that,*

for each $x \in E$, the set $A(x) = \{T\alpha x : \alpha \in I\}$ is bounded in F. Then A is equicontinuous; hence, for any bounded set B in E, $\bigcup\{T\alpha(B) : \alpha \in I\}$ is a bounded set in F.

Theorem 1.2.22. (Open mapping theorem) *Let E of F be complete metrizable TVSś and $T : E \to F$ a continuous and onto linear mapping. Then T is an open mapping. If, in addition T is one to one, then $T^{-1} : F \to E$ is also continuous.*

Theorem 1.2.23. (Closed graph theorem) *Let E and F be complete metrizable TVSś. If $T : E \to F$ is linear and closed (i.e. the graph $G = \{(x, Tx) : x \in E\}$ of T is closed in $E \times F$), then T is continuous.*

1.3. Banach Algebras

In this section, we present basic definitions and results on Banach algebras; for details, see the books of Bonsall and Duncan [3], Rickart [22] and Simmons [25].

Definition 1.3.1. A vector space A over a field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is said to be an *algebra* if, for all $x, y \in A$, the product xy is defined and $xy \in A$, satisfying the following conditions:

(i) $x(yz) = (xy)z = xyz$,

(ii) $x(y+z) = xy + xz$; $(y+z)x = yx + zx$,

(iii) $(\lambda x)(\mu y) = (\lambda\mu)xy$,

where $\lambda, \mu \in \mathbb{K}$ and $x, y, z \in A$.

Definition 1.3.2. An algebra A is called a *commutative algebra* if $xy = yx$ for all $x, y \in A$. An element $e \in A$ is called an *identity* of A if $ex = xe = x$ for all $x \in A$; in this case, A is called an *algebra with identity.*

Definition 1.3.3. A subalgebra I of algebra A is said to be a

(a) *left ideal* if $AI \subseteq I$,

(b) *right ideal* if $IA \subseteq I$,

(c) *two-sided ideal* (or simply, an *ideal*) if $AI \subseteq I$ and $IA \subseteq I$.

Definition 1.3.4. A proper subalgebra $M \subsetneq A$ which is a left (resp. right or two-sided ideal) is said to be a *maximal left (resp. right or two-sided ideal)* if, for any left ideal (resp. right or two-sided) L in A,

$$M \subsetneq L \subseteq A \text{ implies that either } M = L \text{ or } L = A.$$

Definition 1.3.5. Let A be an algebra. Then the set

$$rad(A) = \cap\{M : M \text{ maximal ideal in } A\}$$

is called the *radical* of A. Note that $rad(A)$ is a two-sided ideal in A. An algebra A is called *semisimple* if $rad(A) = \{0\}$; A is called *simple* if it has no two-sided ideal other than $\{0\}$ and A.

Definition 1.3.6. Let A and B be two algebras over the same field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). A linear mapping $\phi : A \to B$ is called an

(i) *algebra homomorphism* if

$$\phi(xy) = \phi(x)\phi(y) \quad \text{for all} \quad x, y \in A;$$

(ii) *algebra anti-homomorphism* if

$$\phi(xy) = \phi(y)\phi(x) \quad \text{for all} \quad x, y \in A;$$

(iii) *isomorphism* if it is algebra homomorphism and one-one.

Definition 1.3.7. An algebra A with a norm $\|.\|$ is called a *normed algebra* if

$$\|xy\| \leq \|x\|\|y\| \quad \text{for all} \quad x, y \in A.$$

Definition. 1.3.8. A complete normed algebra A is called a *Banach algebra.*

Examples 1.3.9. **(a)** For any topological space X, let $C(X)$ be the set of all continuous $\mathbb{K}$-valued functions on X, and let $C_b(X)$ (resp. $C_o(X)$) be the subset of $C(X)$ consisting of those functions which are bounded (resp. vanish at infinity). Then each of the function space $C = C(X), C_b(X)$, or $C_0(X)$ is an algebra over the field $\mathbb{K}$ under the operations

(a) $(f+g)(x) = f(x) + g(x)$ (pointwise addition),

(b) $(\lambda f)(x) = \lambda f(x)$ (pointwise scalar multiplication),

(c) $(fg)(x) = f(x)g(x)$ (pointwise multiplication)

for all $f, g \in \mathcal{C}$, $\lambda \in \mathbb{K}$, and $x \in X$. Further, $C_b(X)$ and $C_0(X)$ are Banach algebras with norm $\|.\|$ defined by

$$\|f\| = \sup\{| f(x) | : x \in X\}, \qquad f \in C_0(X) \text{ or } C_b(X).$$

Note that $C_b(X)$ is an algebra with identity **1**, where $\mathbf{1}(x) = 1$ for all $x \in X$; if X is non-compact, then $C_0(X)$ is an algebra without an identity. The above norm is not well-defined on $C(X)$ since $\|f\|$ does not exist for unbounded $f \in C(X)$.

(b) Let $(E, \|.\|)$ and $(F, \|.\|)$be a normed spaces and $BL(E, F)$ the set of all bounded (or, equivalently) continuous linear mappings $T : E \to F$. Then $BL(E, F)$ is a normed space with pointwise addition and scalar multiplication and operator norm:

$$\|T\| = \sup\{\|T(x)\| : x \in E, \|x\| \leq 1\}.$$

It is a Banach space if F is so. If $F = E$,then $BL(E) = BL(E, E)$ is a normed algebra with composition multiplication:

$$(ST)(x) = S(T(x)), \qquad S, T \in BL(E),$$

and the norm as define above.

Definition 1.3.10. Let A be an algebra with identity e. For any $a, b \in A$, b is called the *inverse* of a if $ab = e = ba$. In this case, a is called invrtible (or regular). An element $c \in A$ is called *singular* if it is not invertible.

Theorem 1.3.11. *Let A be a normed algebra with identity e, and let G_A be the set of all invertible elements of A. Then:*

(a) The map $a \to a^{-1}$ of G_A into G_A is homeomorphism.

(b) G_A is open in A.

(c) G_A is a topological group.

Definition 1.3.12. Let A be an algebra with identity e. For any $a \in A$, we define the *spectrum* of a by

$$sp(a) = sp_A(x) = \{\lambda \in \mathbb{C} : \lambda e - a \text{ is singular }\}.$$

.**Theorem 1.3.13.** *Let A be a Banach algebra with identity e. Then, for any $a \in A$,*

(a) $sp(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|$; hence $sp(a)$ is a bounded subset of $\mathbb{C}$.

(b) $sp(a)$ is closed and hence is compact.

(c) $sp(a) \neq \phi$.

Definition 1.3.14. *Let A be a normed algebra with identity e. For any $a \in A$, the spectral radius of a is defined as*

$$r(a) = inf\ \{\|x^n\|^{\frac{1}{n}} : n = 1, 2, ...\}.$$

Theorem 1.3.15. *Let A be a normed algebra with identity e. For any $a \in A$,*

(a) $0 \leq r(a) \leq \|a\|$.

(b) $r(a) = \lim\limits_{n \to \infty} \|a^n\|^{\frac{1}{n}} = sup\{|\lambda| : \lambda \in sp(a)\}$.

Definition 1.3.16. Let A be a normed algebra without identity. Then the *unitization* of A is the normed algebra. $A_u = A \times \mathbb{K} = \{(x, \alpha) : x \in A, \alpha \in \mathbb{K}\}$ with the operations

$$\begin{aligned} (x, \alpha) + (y, \beta) &= (x + y, \alpha + \beta) \\ \beta(x, \alpha) &= (\beta x, \beta\alpha) \\ (x, \alpha)(y, \beta) &= (xy + \alpha y + \beta x, \alpha\beta) \end{aligned}$$

and the norm $\|(x, \alpha)\| = \|x\| + |\alpha|$. In this case, $e_1 = (0, 1)$ is the identity of A_u and $\|e_1\| = 1$. Further, the mapping $a \to (a, 0)$ is an isometric isomorphism of A onto a subalgebra $A_0 = \{(x, 0) : x \in A\}$ of A_u.

Definition 1.3.17. Let A be an algebra. For any $x, y \in A$, the *quasi-product* $x * y$ of x, y is defined by

$$x * y = x + y - xy.$$

Definition 1.3.18. Let A be a an algebra and $x, y, z \in A$.Then

(i) y is called a *left quasi-inverse* of x if $y * x = 0$.

(ii) z is called a *right quasi-inverse* of x if $x * z = 0$.

(iii)y is called a *quasi-inverse* of x if $y * x = 0 = x * y$;

Definition 1.3.19. Let A be an algebra and $x \in A$.

(i) x is called *left quasi-invertible* if x has a left quasi-inverse. Similarly, we define *right quasi-invertible* and *quasi-invertible* elements in A.

(ii) x is called *quasi-singular* if it is not quasi-regular.

Theorem 1.3.20. *Let A be a Banach algebra with identity e.*

(i) If $x \in A$ with $r(x) < 1$ (or $\|x\| < 1$), then $e - x$ is invertible and

$$(e-x)^{-1} = \sum_{n=0}^{\infty} x^n = e + \sum_{n=0}^{\infty} x^n.$$

(Here $x^0 = e$.)

(ii) If $x \in A$ with $r(e-x) < 1$ (or $\|e - x\| < 1$), then x is invertible and

$$x^{-1} = [e-(e-x)]^{-1} = \sum_{n=0}^{\infty} (e-x)^n.$$

(iii) If $x \in A$ and $\lambda(\neq 0) \in \mathbb{C}$ with $\|x\| < |\lambda|$, then $\lambda e - x$ is invertible and

$$(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}.$$

Theorem 1.3.21. *Let A be any Banach algebra. If $x \in A$ with $r(x) < 1$, then x is quasi-invertible and its quasi-inverse x_{-1} is given by*

$$x_{-1} = -\sum_{n=0}^{\infty} x^n.$$

Theorem 1.3.22. *Let A be an algebra and suppose $x \in A$. Then the following are equivalent:*

(i) x has a left (right) quasi-inverse.

(ii) If $M \subseteq A$ is maximal modular left (right) ideal, then $\exists$ some $y \in A$ such that $y * x \in M$ ($y * x \in M$).

Definition 1.3.23. *Let A be a complex algebra and let $x \in A$.*

(i) If A has an identity, the spectrum $sp(x)$ of x, as defined before, is

$$sp(x) = sp_A(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is singular}\}.$$

(ii) If A is without identity, the spectrum $sp(x)$ of x is defined by

$$sp(x) = sp_A(x) = \{0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq 0, \frac{x}{\lambda} \text{ is quasi- singular}\}.$$

Theorem 1.3.24. *Let A be an algebra without identity. Then, for any $x \in A$,*

$$sp_A(x) = sp_{A_u}((x,0)).$$

Theorem 1.3.25. (Gelfand-Mazur) *Let A be a complex Banach algebra with identity e. If A is a division algebra (i.e. if every non-zero $x \in A$ is invertible), then $A = \{\lambda e : \lambda \in \mathbb{C}\}$; hence $A \cong \mathbb{C}$.*

Definition 1.3.26. Let A be an algebra.

(a) A left (resp. right) ideal I in A is called a *left* (resp. *right*) *modular ideal* if there exists an element $u \in A$ such that

$$x - ux \in I \text{ (resp. } x - xu \in A) \text{ for all } x \in A.$$

In this case, u is called a *left* (*right*) *modular unit* for A.

(b) A two-sided ideal I in A is called a *modular ideal* if it is both a left and right modular ideal.

(c)A (left, right,two-sided) ideal I in A is called *maximal modular* if it is proper and is not contained in any other such ideal.

Clearly, if A has an identity e, then every ideal in A is modular with e a modular unit.

Definintion 1.3.27. Let A be a complex Banach algebra. A linear functional ϕ on A is called a *multiplicative linear functional* (in short *m.l.f*) if

(i) ϕ is non-zero on A (i.e $\phi(x) \neq 0$ for some $x \in A$).

(ii) $\phi(xy) = \phi(x)\,\phi(y)$ for all $x, y \in A$.

Theorem 1.3.28. *Let ϕ be a m.l.f on a complex Banach algrbra A. Then*

(i) ϕ is continuous on A and $\|\phi\| \leq 1$.

(ii) If A has an identity e, then $\phi(e) = 1$; if $\|e\| = 1$, then $\|\phi\| = 1$.

Note : A m.l.f. ϕ need not be continuous on a non-complex normed algebra.

Theorem 1.3.29. *Let A be a commutative Banach algebra.*

(i) If ϕ is a m.l.f. on A, then $M = \phi^{-1}(0) = ker\ \phi$ is a maximal modular ideal in A.

(ii) If M is a maximal modular ideal in A, then $\exists$ a unique m.l.f. ϕ on A such that $M = ker\phi$.

Notation. For any commutative Banach algebra A, let

$\Phi(A)$ = the set of all maximal modular ideals in A.

$\Delta(A)$ = the set of non-zero multiplicative linear functinals on A.

Note. By the above Theorem, there is a one-one correspondence between $\Phi(A)$ and $\Delta(A)$ given by $M \leftrightarrow \phi$, where $M = \phi^{-1}(0)$. We mention that $\Delta(A) \subseteq A^*$, but need not be a vector subspace.

Theorem 1.3.30. *Let A be a commutative Banach algebra without identity and let A_u be the unitization of A. Let $\phi_0 \in \Delta(A_u)$ such that $\phi_0 = 0$ on A. Then $\Delta(A_u) = \Delta(A) \cup \{\phi_0\}$.*

Note. Let A be a commutative Banach algebra. Then, for any $\phi \in \Delta(A)$, $\phi \in A^*$ and $\|\phi\| \leq 1$. Hence $\Delta(A) \subseteq$ norm-closed unit ball in A^*.

Definition 1.3.31. Since $\Delta(A) \subseteq A^*, \Delta(A)$ can be given the relative weak*–topology $\sigma(A^*, A)$ of A^*. This topology on $\Delta(A)$ is usually called the *Gelfand topology*.

Recall that by the Banach Alaoglu theorem, the closed unit ball in A^* is weak*-compact.

Theorem 1.3.32. *Let A be a commutative Banach algebra .*

(i) If A has an identity e, then $\Delta(A)$ (with the Gelfand topology) is a compact Hausdorff space.

(ii) If A has no identity, then $\Delta(A)$ is a locally compact Hausdorff space with one-point compactification $\Delta(A) \cup \{\phi_0\}$, where $\phi_0 = 0$, the zero functional on A.

Theorem 1.3.33. (Gleason-Kahane-Zelazko). *Let A be a Banach algebra and ϕ a linear functional on A. Then the following two conditions are equivalent:*

(i) ϕ is multiplicative.

(ii) $\phi(a) \in sp(a)$ for all $a \in A$.

Theorem 1.3.34. *Let A be a commutative Banach algebra, $a \in A$ and $\lambda \in \mathbb{C}$.*

(i) If $\lambda \neq 0$, then

$$\lambda \in sp(a) \Leftrightarrow \lambda = \phi(a) \text{ for some m.l.f. } \phi \text{ on } A.$$

(ii) If A has an identity e, Then (i) holds for $\lambda = 0$; hence

$$sp(a) = \{\phi(a) : \phi \in \Delta(A)\}.$$

Theorem 1.3.35. *In every commutative Banach algebra with identity, there exists at least one multiplicative linear functional.*

Note: $\Delta(A)$ may be empty in the following cases:

(a) A is a commutative Banach algebra without an identity.

(b) A is non-commutative Banach algebra with identity.

Definition 1.3.36. For each $x \in A$, define $\hat{x}: \Delta(A) \to \mathbb{C}$ by

$$\hat{x}(\phi) = \phi(x), \ \phi \in \Delta(A)$$

Then $\hat{x}$ is continuous and bounded on $\Delta(A)$; hence $\hat{x} \in C(\Delta(A))$. The function $\hat{x}$ is called the *Gelfand transform* of x on A. Setting $\hat{A} = \{\hat{x}: x \in A\}$, we have $\hat{A} \subseteq C(\Delta(A))$. The map $\Psi : x \to \hat{x}$ is called the *Gelfand representation* of A into $C(\Delta(A))$.

Theorem 1.3.37. *Let A be a commutative Banach algebra.*

(a) $\left\|\hat{x}\right\|_\infty \leq \|x\|$ *for all* $x \in A$.

(b) The map $\Psi : x \to \hat{x}$ *is an algebaric homomorphism of* A *onto the subalgebra* $\hat{A}$ *of* $C_0(\Delta(A))$.

(c) Ψ *is one-one* $\Leftrightarrow$ A *is semi-simple.*

(d) $\hat{A}$ *separates points of* $\Delta(A)$.

(e) If A *has an identity* e, *then* $\hat{A}$ *contains all the constant functions.*

Definition 1.3.38 (a) An *involution* in an algebra A is a mapping $x \to x^*$ of A into itself with the following properties:

(i) $(x+y)^* = x^* + y^*$,

(ii) $(\lambda x)^* = \overline{\lambda}\, x^*$,

(iii) $(x^*)^* = x$,

(iv) $(xy)^* = y^* x^*$

for all $x, y \in A$ and $\lambda \in \mathbb{K}$. In this case A is called a *$*$-algebra.*

(b) A Banach algebra A with involution is called a *Banach $*$-algebra.*

(c) A Banach *-algebra is called a *C^*-algebra* (or *B^*-algebra*) if $\|xx^*\| = \|x\|^2$ for all $x \in A$.

Note: In some books, a C^*-algebra is defined as any closed self-adjoint subalgebra of $B(H)$ for some Hilbert space H and then it is proved that every B^*-algebra is a C^*-algebra.

Theorem 1.3.39. (Gelfand-Naimark) *Let A be a commutative C *-algebra and $\Delta(A)$ the maximal ideal space of A. Then:*

(a) If A has an identity, then A is isometrically $*$-isomorphic to $C(\Delta(A))$ with $\Delta(A)$ a compact Husdorff space.

(b) If A is without identity , then A is isometrically $*$-isomorphic to $C_0(\Delta(A))$ with $\Delta(A)$ a locally compact Hausdorff space.

We now describe the **hull-kernel topology** on the modular maximal ideal space. Let A be a commutative Banach algebra with $\Delta(A)$ its the maximal ideal space.

Definition 1.3.40. For any $E \subseteq \Delta(A)$, the kernel of E is defined by

$$k(E) = \cap\{ker\ \phi : \phi \in E\}$$

Note that $k(E)$ is a closed ideal in A.

Definition 1.3.41. For any $J \subseteq A$, the hull of J is defined by

$$\begin{aligned} h(J) &= \{\phi \in \Delta(A) : J \subseteq ker\ \phi\} \\ &= \bigcap_{a \in J} \{\phi \in \Delta(A) : \hat{a}\ (\phi) = 0\} \end{aligned}$$

Note Since each $\hat{a}$ is continuous, the set $\{\phi \in \Delta(A) : \hat{a}\ (\phi) = 0\}$ is a closed subspace of $\Delta(A)$. Hence $h(J)$ is a closed subset of $\Delta(A)$ in the w^*or Gelfand topology.

Definintion 1.3.42. For any $E \subseteq \Delta(A)$, the *hull-kernel closure* of E is defined by

$$\bar{E} = h(k(E)).$$

Theorem 1.3.43. *Let A be a commutative Banach algebra.*

(i) For any $E \subseteq \Delta(A), k(E)$ is a closed ideal in A and $kh(k(E)) = k(E)$

(ii) If J is an ideal in A, then $hkh(J) = h(J)$.

Theorem 1.3.44. *Let A be a commutative Banach algebra. The function $E \to \bar{E}$ from $P(\Delta(A)) \to P(\Delta(A))$ is a Kuratowski closure operation on $\Delta(A)$; $i.e.$*

(i) $\bar{\phi} = \phi$, $\overline{\Delta(A)} = \Delta(A)$.

(ii) $E \subseteq \bar{E}$.

(ii) $\overline{(\bar{E})} = \bar{E}$

(iv) $\overline{(E_1 \cup E_2)} = \bar{E}_1 \cup E_2$.

Definition 1.3.45. Let $E \to \bar{E}$ be the closure operation defined above. Then

$$\tau_{hk} = \{X \setminus \bar{E} : E \subseteq \Delta(A)\}$$

is a topology on $\Delta(A)$, called the *hull-kernel topology*.

Theorem 1.3.46. *Let A be a commutative Banach algrbra.*

(i) $\tau_{hk} \leq w^*$ *on* $\Delta(A)$

(ii) If A has an identity, then $(\Delta(A), \tau_{hk})$ is compact.

(iii) τ_{hk} is Hausdorff $\Leftrightarrow \tau_{hk} = w^*$ on $\Delta(A)$.

Definition 1.3.47. A commutative Banach algebra A is said to be *regular* if $\tau_{hk} = w^*$ on $\Delta(A)$.

Theorem 1.3.48. *If A is a regular commutative Banach algebra then*

$$\partial A = \Delta(A).$$

We next introduce Shilov boundary. Let X be a locally compact Hausdorff space and $\mathcal{C} \subseteq C_0(X)$. It follows that if $F \subseteq X$ a closed set and $f \in \mathcal{C}$, then $\exists\, x_0 \in F$ such that

$$\sup_{x \in F} |f(x)| \leq |f(x_0| \quad (i.e.\ |f(x)| \leq |f(x_0)| \quad \text{for all } x \in F).$$

Definition 1.3.49. Let X be a locally compact Hausdorff space and $\mathcal{C} \subseteq C_0(X)$. A closed subset F of X is called a maximizing set for $\mathcal{C}$ if

$$\sup_{x \in F} |f(x)| = \|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Let $\partial_{\mathcal{C}} X$ be the intersection of all maximizing sets for $\mathcal{C}$. Then $\partial_{\mathcal{C}} X$ is called the *Shilov boundary* of $\mathcal{C}$ in X. Clearly, $\partial_{\mathcal{C}} X$ is the unique minimal closed maximizing set for $\mathcal{C}$.

Theorem 1.3.50. *Let $\mathcal{C}$ be a subalgebra of $C_0(X)$ which separates the points of X. Then the Shilov boundary $\partial_{\mathcal{C}} X$ exists.*

Note As a particular case, let A be a commutative Banach algebra. Then the maximal modular ideal space $\Delta(A)$(or $\Phi(A)$) is locally compact Hausdorff in the Gelfand topology. Further, $\hat{A} = \{\hat{x}\colon x \in A\}$ is a subalgebra of $C_0(\Delta(A))$ such that $\hat{A}$ separates the points of $\Delta(A)$. Hence, by the above theorem, the Shilov boundary $\partial_{\hat{A}}\ \Delta(A)$ exists. i.e.

$$\sup_{m\in\partial_{\hat{A}}\Delta(A)} \left|\hat{x}(m)\right| = \left\|\hat{x}\right\|_{\infty} = \sup_{m\in\Delta(A)} \left|\hat{x}(m)\right|$$

In this case, $\partial\Delta(A) = \partial_{\hat{A}}\ \Delta(A)$ is called the *Shilov boundary* of A.

1.4 Topological Algebras

Now we give some definitions and results from topological algebras; for detail, see the books of Mallios [20] and Husain [11], and the monograph [9] of Fragoulopoulou..

Definition 1.4.1. An algebra A over $\mathbb{K}$ with a topology τ is called a *topological algebra* (resp. *semitopological algebra*) if it is a topological vector space in which multiplication is jointly (resp. separately) continuous. A complete metrizable topological algebra is called an *F-algebra.* If a semitopological algebra is complete and metrizable, then it is a topological algebra.

Theorem 1.4.2. If (A, τ) is a topological algebra, there exists a base $\mathbb{W}$ of τ-neighbourhoods of 0 consisting of closed, balanced and absorbing sets such that for each $G \in \mathbb{W}$, there exists an $H \in \mathbb{W}$, with $H^2 \subseteq G$. (Here $H^2 = HH = \{ab : a, b \in H\}$).

Definition 1.4.3. A subset H of an algebra A is called *idempotent* if $H^2 \subseteq H$.

Definition 1.4.4. Let A be a topological algebra. Then A is said to be a

(a) *locally convex algebra* if it has a base $\mathbb{W}$ of neighbourhoods of 0 consisting of convex sets;

(b) an *F normed algebra* if it is Hausdorff and has a countable base $\mathbb{W}$ of neighbourhoods of 0 consisting of idempotent sets;

(c) *locally m-convex algebra* if it has a base $\mathbb{W}$ of neighbourhoods of 0 consisting of convex and idempotent sets.

Every normed algebra and, more generally, every locally m-convex algebra is locally convex algebra. Further, every Hausdorff locally bounded algebra is metrizable.

Theorem 1.4.5. Let A be a Hausdorff topological algebra.

(a) If A is locally convex, then its topology is generated by a family of seminorms.

(b) If A is an F-normed algebra, then its topology is generated by a single submultiplicative F-norm.

(c) If A is locally m-convex, then its topology is generated by a family of submultiplicative seminorms.

Definition 1.4.6. (a) A complete F-normed algebra is called an *F-algebra*

(b) A locally m-convex F-algebra is called a *Fréchet algebra.*

(c) A locally convex F-algebra is called a *B_0 algebra.*

Remarks. If H is an idempotent neighbourhood of 0 in a topological algebra A, then it is easily seen that its Minkowski functional ρ_H is submultiplicative: $\rho_H(xy) \leq \rho_H(x)\rho_H(y)$ for all $x, y \in A$. Further, if A has an identity e, $\rho_H(e) \geq 1$.

Definition 1.4.7. A topological algebra A with identity is said to be a *Q-algebra* if the set of all its invertible elements is open.

Every Banach algebra is a Q-algebra. On the other hand, $C(\mathbb{R})$ with the compact- open topology k is a complete metrizable locally m-convex algebra but not a Q-algebra.

Theorem 1.4.8. *(a) The closure of every proper (left, right or two-sided) ideal in a Q-algebra is a proper (left, right, or two-sided) ideal.*

(b) Every maximal ideal (left, right or two-sided) in a Q-algebra with identity is closed.

Definition 1.4.9. A net $\{e\lambda : \lambda \in I\}$ in a topological algebra A is called a

(a) *left approximate identity* if

$$\lim_{\lambda} e\lambda x = x \qquad \text{for all } x \in A;$$

(b) *right approximate identity* if

$$\lim_{\lambda} x e\lambda = x \qquad \text{for all } x \in A;$$

(c) *two-sided approximate identity* if

$$\lim_{\lambda} e\lambda x = \lim_{\lambda} xe\lambda = x \qquad \text{for all } x \in A.$$

Definition 1.4.10 If A is a locally m-convex algebra whose topology is generated by a family $\{p\alpha : \alpha \in J\}$ of submultiplicative seminorms, then an approximate identity $\{e\lambda : \lambda \in I\}$ in A is called *uniformly bounded* if

$$\sup \{p\alpha(e\lambda) : \alpha \in J, \lambda \in I\} < \infty.$$

Definition 1.4.11. An algebra A is said to be *faithful* (or *without order or proper or without annihilators*) if aA or $Aa = \{0\}$ implies that $a = 0$.

Theorem 1.4.12. *An algebra is faithful in each of the following cases:*

(1) A has an identity;

(2) A is a topological algebra with an approximate identity (e.g. A is a B^*-algebra)

(3) A is a topological algebra with an orthogonal basis;

(4) A is a commutative semisimple Banach algebra.

Definition 1.4.13. Let A be a topological algebra. Then A is called

(a) *factorable* if $A = A^2$ (i.e. for each $x \in A$, there exist $y, z \in A$ such that $x = yz$);

(b) *strongly factorable* if, for any sequence $\{x_n\}$ in A with $x_n \to 0$, there exists $y \in A$ and a sequence $\{z_n\}$ in A with $z_n \to 0$ such that $x_n = yz_n$ for all $n \geq 1$.

Clearly, strong factorability implies factorability.

We next state some important generalizations of the famous Cohen's factorization theorem, due to Craw [6] (see also Johnson [16], Doran and Wichmann [7], Husain [11]).

Theorem 1.4.14. *Let A be a Fréchet algebra with a uniformly bounded left approximate identity and submultiplicative seminorms. Then A is strongly factorable, in particular, factorable.*

Note that the above theorem is a so-called *left-hand version* of the factorization theorem. Clearly, its *right-hand version* also holds.

1.5. Bases in Banach spaces and Topological vector spaces

Definiton 1.5.1. Let E be a Hausdorff TVS. A sequence $\{x_n\}$ in E is called a *basis* of E if, for each $x \in E$, there exists a unique sequence $\{\lambda_n\}$ of scalars such that

$$x = \lim_{k\to\infty} \sum_{i=1}^{k} \lambda_i x_i \ ;$$

that is, for any neighbourhood U of 0 in E, there exists an integer $N \geq 1$ such that for all $k \geq N$

$$\sum_{i=1}^{k} \lambda_i x_i - x \in U.$$

In this case we write $x = \sum_{i=1}^{\infty} \lambda_i x_i$.

Note. (1) λ_n 's are real or complex numbers depending on whether E is a real or complex vector space.

(2) If $E = (E, \|.\|)$ is a normed space , then $x = \sum_{n=1}^{\infty} \lambda_n x_n$ means that

$$\left\| x - \sum_{i=1}^{k} \lambda_i x_i \right\| \to 0, k \to \infty;$$

that is, given any $\varepsilon > 0, \exists\, N \geq 1$ such that $\left\| X - \sum_{i=1}^{k} \lambda_i x_i \right\| < \varepsilon$ for all $k \geq N$.

Definition 1.5.2. Let $\{x_n\}$ be a basis of E.

(1) For each $m = 1, 2, ...$ define $f_m : E \to \mathbb{K}$ by $f_m(x) = \lambda_m$ for $x = \sum_{i=1}^{\infty} \lambda_i x_i \in E$; *that is,* $f_m(\sum_{i=1}^{\infty} \lambda_i x_i) = \lambda_m$. Then f_m are called the *associated cofficient functionals* of the basis$\{x_n\}$. We shall verify below that f_m are linear and

$$f_m(x_n) = \delta_{mn} = \{ \begin{matrix} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{matrix},$$

but f_m may or may not be continuous.

(2) If all f_m are continuous, the basis $\{x_n\}$ is called a *Schauder basis.*

Lemma 1.5.3. *Let* $\{x_n\}$*be a basis of* E *with* $\{f_n\}$*the sequance of associated coefficient functionals. Then:*

(a) Each f_m is linear.

(b) $f_m(x_n) = \delta_{mn}$

Proof. (a) Let $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. Since $\{x_n\}$is a basis, we can write x and y uniquely as

$$x = \sum_{i=1}^{\infty} \lambda_i x_i, \qquad y = \sum_{i=1}^{\infty} \mu_i x_i.$$

Clearly, $f_m(x) = \lambda_m, f_m(y) = \mu_m$.

Now, by continuity of vector addition and scalar multipliction in E,

$$\begin{aligned}
\alpha x + \beta y &= \alpha \sum_{i=1}^{\infty} \lambda_i x_i + \beta \sum_{i=1}^{\infty} \mu_i x_i \\
&= \alpha \lim_{k\to\infty} \sum_{i=1}^{k} \lambda_i x_i + \beta \lim_{k\to\infty} \sum_{i=1}^{k} \mu_i x_i \\
&= \lim_{k\to\infty} \alpha \sum_{i=1}^{k} \lambda_i x_i + \lim_{k\to\infty} \beta \sum_{i=1}^{k} \mu_i x_i \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} (\alpha\lambda_i + \beta\mu_i) x_i \\
&= \sum_{i=1}^{\infty} (\alpha\lambda_i + \beta\mu_i) x_i.
\end{aligned}$$

Hence

$$f_m(\alpha x + \beta y) = \alpha\lambda_m + \beta\mu_m = \alpha f_m(x) + \beta f_m(y);$$

that is , f_m is linear.

(b) For any $n \geq 1$, we can write

$$x_n = \sum_{i=1}^{\infty} \lambda_i x_i = \sum_{i=1}^{\infty} f_i(x_n) x_i$$

Also $x_n = 1.x_n$. By uniqueness of representation,

$$f_m(x_n) = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}.$$

■

Examples of Bases. (1) For each $n \geq 1$, let $e_n = (\delta_{nm} : n = 1, 2, ...) = (0, 0, ..., 0, 1, 0, ...)$ with 1 only at the nth place. Then $\{e_n\}$ is a basis for the normed spaces c_0 and $l_p\,(1 \leq p < \infty)$ of sequences, but not for c and l_∞. (Recall that $c_0 \subseteq c \subseteq l_\infty$.). [If $x = (a_n) \in l_p$ or c_0, taking $\{\lambda_n\} = \{a_n\}$, we have

$$\lim_{k\to\infty} \sum_{i=1}^{k} \lambda_i e_i = \lim_{k\to\infty} (a_1, ...a_k, 0, 0, ...) = x.]$$

(2) The normed space l_∞ has no basis. In fact, we shall later show that every TVS with a basis is separable. Since l_∞ is not separable it can not have a basis.

(3) Consider $E = c$ and let $u = (1, 1, 1, ...) \in c$. Then $\{u, e_1, e_2, ...\}$ is a basis for c.In fact, if $x = (a_n) \in c$ with $a_n \to a \in \mathbb{K}$, then we can write

$$x = au + (a_1 - a)e_1 + (a_2 - a)e_2 + ...$$

(4) Every separable Hilbert space has a basis.

(5) If X is a compact metric space, then $(C(X), \|\cdot\|_\infty)$ has a basis.

Examples of Schauder Bases. (1) The normed spaces $C[0,1]$ and $C^k[0,1]$ have Schauder basis but no unconditional basis.

(2) Every basis in a complete metrizabale TVS (in particular,Banach space) is a Schauder basis.

(3) Any TVS E with $\acute{E} = \{0\}$ $(e.g.\ E = L_p, 0 < p < 1)$ can not have a Schauder basis (since there exist no non-zero continuous linear functionals on E).

Theorem 1.5.4. *Let E be a Hausdorff TVS. If $\{x_n\}$ is a basis for E, then $\{x_n\}$ is a linearly independent set. The converse is not true.*

Proof. For any countable subset $\{\alpha_n\}$ (finite or infinite) of $\mathbb{K}$, suppose

$$\sum \alpha_i x_i = 0.$$

Since $\{x_n\}$ is a basis , it follows from the uniqueness of α_i, s that $\alpha_i = 0$ for all i. Hence $\{x_n\}$ is linearly independent. ■

Example. Let $E = c_0$ or l_1. Take

$$x_1 = (1/2)e_1, \;\; x_2 = (1/2)e_2 - e_1, ..., x_n = (1/2)e_n - e_{n-1}, ...$$

Then

(i) $\{x_n\}$ is linearly independent.

(ii) $\sum\limits_{n=1}^{\infty} (1/2^n)x_n = 0$ and so $\{x_n\}$ is not a basis for E (since the coefficients $1/2^n$ are non-zero).

Solution. (i) Let $\{x_{n1}, x_{n2}, ...x_{nk}\}$ be any finite subset of $\{x_n\}$ and suppose, for any $\alpha_1, \alpha_2, ...\alpha_k \in \mathbb{K}$.

$$\sum_{i=1}^{k} \alpha_i x_{n_i} = 0. \qquad 0.1$$

Since each x_{n_i} is a finite linear combination of e_n, s , we can express (1) as

$$\sum_{i=1}^{k} \alpha_i \beta_i e_i = 0,$$

where each $\beta_i \neq 0$. Since $\{e_n\}$, being a basis for E, is linearly independent , we have $\alpha_i = 0$ for all $i = 1, 2, ...k$. Hence $\{x_n\}$ is linearly independent.

(ii) For any $k > 1$,

$$\begin{aligned}\sum_{n=1}^{k} (1/2^n)x_n &= (1/2)((1/2)e_1) + (1/2^2)((1/2)e_2 - e_1) + (1/2^3)((1/2)e_3 - e_2) + ...\\ &\quad +(1/2^k)((1/2)e_k - e_{k-1})\\ &= (1/2^2)e_1 + ((1/2^3)e_2 - (1/2^2)e_1) + ((1/2^4)e_3 - (1/2^3)e_2) + ...\\ &\quad +((1/2^{k+1})e_k - (1/2^k)e_{k-1})\\ &= (1/2^{k+1})e_k = (0, 0, ...0, 1/2^{k+1}, 0, ...)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=1}^{k} (1/2^n)x_n &= \lim_{k\to\infty}\sum_{n=1}^{k} (1/2^n)x_k = \lim_{k\to\infty} (1/2^{k+1})e_k\\ &= \lim_{k\to\infty} (0, 0, ...0, 1/2^{k+1}, 0, ...) = (0, 0, ...) = 0.\end{aligned}$$

Since the coefficients $1/2^n$ are non-zero, $\{x_n\}$ cannot be a basis for E. ■

Remark. *If $\{x_n\}$ is a basis of a normed space E then, without loss of generality, it may be assumed that* $\|x_n\| = 1$ *for all* $n \geq 1$. In fact, if $x \in E$ and $x = \sum_{i=1}^{\infty} \lambda_i x_i$,then we can write

$$x = \sum_{i=1}^{\infty} \|x_i\| \lambda_i (x_i / \|x_i\|) = \sum_{i=1}^{\infty} \acute{\lambda}_i y_i,$$

where $\acute{\lambda}_i = \|x_i\| \lambda_i$ and $y_i = x_i / \|x_i\|$ with $\|y_i\| = 1$. In other words, a basis $\{x_n\}$ of a normed space can be *normalized* by assuming that $\|x_n\| = 1$ for all $n \geq 1$.

Note. (1) A basis is rarely a Hamel basis.

(2) If a TVS E has a basis which is also a Hamel basis, then E would have a countable Hamel basis and would thus be ϕ with a suitable topology.In fact, a Housdorff TVS which has a countable infinite Hamel basis must be of first category. In particular, a complete metrizable TVS connot have a countable infinite Hamel basis (since it is of second category).

Theorem 1.5.5. *Any TVS E with a basis $\{x_i\}$ is separable. (Hence a non-separable TVS (e.g. l_∞) has no basis.)*

Proof. Let $B = \{\sum_{i=1}^{k} r_i x_i : r_i \text{ rational}, k \geq 1\}$. Clearly B is a countable subset of E. To show that B is dense in E, let $x \in E$ and W any neighbourhood of 0 in E. Choose a neighbourhood V of 0 in E such that $V + V \subseteq W$. Without any loss of generality, we may assume that E is a real vector space. Since $\{x_i\}$ is a basis, there exists a unique sequence $\{\lambda_i\} \subseteq \mathbb{R}$ such that

$$x = \sum_{i=1}^{\infty} \lambda_i x_i = \lim_{k\to\infty} \sum_{i=1}^{k} \lambda_i x_i.$$

Then there exists an integer $N \geq 1$ such that for all $n \geq N$

$$\sum_{i=1}^{k} \lambda_i x_i - x \in V.$$

Choose a balanced neighbourhood U of 0 in E such that

$$U + U + ... + U(N - terms) \subseteq V.$$

There exists $M > 0$ such that $\{x_i, x_i, ...x_n\} \subseteq MU$. For each $i = 1, 2, ...N$, choose a rational r_i with

$$|r_i - \lambda_i| < 1/M.$$

Now $\sum_{i=1}^{N} r_i x_i \in B$. Further

$$\begin{aligned} \sum_{i=1}^{N} r_i x_i - x &= \sum_{i=1}^{N} (r_i - \lambda_i) x_i + \sum_{i=1}^{N} \lambda_i x_i - x \\ &\in (1/M)MU + ... + (1/M)MU(N - terms) + V \\ &\subseteq V + V \subseteq W. \end{aligned}$$

Therefore B is dense in E. Hence E is separable. ■

Note. *(Basis Problem)* By above theorem, a non-separable TVS has no basis. In 1932, S. Banach asked the following question.

Does every separable Banach space have a (Schauder) basis?

This remained unsolved for a long time since almost all known separable Banach spaces possess a Schauder basis. However, in 1972, the Swedish mathematician P. Enflo was able to construct a complicated example of a separable Banach space which has no basis.

Definition 1.5.6. A TVS E is called *semicomplete* if every Cauchy sequence in E converges in E.

Theorem 1.5.7. *Let (E,τ) be a semicomplete LCS with a basis $\{x_n\}$, and let $\{p\alpha\}$ be a family of seminorms which generate the topology τ. For each $p\alpha$ and $x=\sum_{i=1}^{\infty}\lambda_i x_i \in E$, put*

$$\acute{P}\alpha(x) = \sup_{k\geq 1} p\alpha(\sum_{i=1}^{k}\lambda_i x_i).$$

Let τ' be the locally convex topology on E generated by the family $\{\acute{P}\alpha\}$. Then :

(a) $\tau\subseteq\tau'$ and (E,τ') is semicomplete and Hausdorff.

(b) If (E,τ) is a complete metrizable LCS, then $.\tau=\tau'$.

Proof. (a) It is easy to see that each $\acute{P}\alpha$ is a seminorm on E and so the family $\{\acute{P}\alpha\}$ generates a locally convex topology τ' on E. For any $p\alpha$ and $x=\sum_{i=1}^{\infty}\lambda_i x_i\in E$

$$\begin{aligned} p\alpha(x) &= p\alpha(\sum_{i=1}^{\infty}\lambda_i x_i) = \lim_{k\to\infty} p\alpha(\sum_{i=1}^{k}\lambda_i x_i) \\ &\leq \sup_{k\geq 1} p\alpha(\sum_{i=1}^{k}\lambda_i x_i) = \acute{P}\alpha(x). \end{aligned}$$

Hence it follows that $\tau\subseteq\tau'$. Since τ is Hausdorff and $\tau\subseteq\tau'$, τ' is also Hausdorff.

To show that (E,τ') is semicomplete, let $\{y_n\}$ be a τ'-Cauchy sequence in E. since $\{x_n\}$ is a basis for (E,τ), we can write each y_n as

$$y_n = \tau-\lim_{k\to\infty}\sum_{i=1}^{k}\lambda_{ni}x_i = \sum_{i=1}^{\infty}\lambda_{ni}x_i.$$

Then, for each α,

$$p\alpha(y_n-y_m)\leq \acute{P}\alpha(y_n-y_m)\to 0 \text{ as } n,m\to\infty$$

as $n,m\to\infty$. Hence $\{y_n\}$ is also a $\tau'-$Cauchy sequence in E. Since E is semicomplete,

$$y=\tau-\lim_{n\to\infty} y_n\in E.$$

Now using the standard argument, we can show that for any α, $\acute{P}\alpha(y_n - y) \to 0$ as $n \to \infty$. Therefore $y_n \xrightarrow{\tau'} y \in E$, and so (E, τ') is semicomplete.

(b) Suppose (E, τ) is a complete metrizable LCS. Then τ is generated by a countable family $\{p_m\}$ of seminorms.Therefore the countable family $\{\acute{P}_m\}$ generate ametrizable locally convex topology τ' on E. Since (E, τ) is complete, using the argument of part(a), it follows that (E, τ') is also complete. Since $\tau \subseteq \tau'$, the identity mapping $i : (E, \tau') \to (E, \tau)$ is continuous. Then , by a corollary of the open mapping theorem,$i = i^{-1} : (E, \tau) \to (E, \tau')$ is continuous. This implies that $\tau' \subseteq \tau$. Thus $\tau = \tau'$. ■

Theorem 1.5.8. *Each basis in a complete metrizable LCS (E, τ) is a Schauder basis.*

Proof. Let $\{p_m\}$ be a countable family of seminorms which generate the topology τ. For each P_m and $x = \sum_{i=1}^{\infty} \lambda_i x_i \in E$, define

$$\acute{P}_m(x) = \sup_{k \geq 1} p_m(\sum_{i=1}^{k} \lambda_i x_i).$$

Then , by last theorem, each $p_m \leq \acute{P}_m$ and that the famliy $\{ \acute{P}_m\}$ generates a topology τ' on E with $\tau' = \tau$.

To show that $\{x_n\}$ is a Schauder basis for (E, τ), let $\{f_n\}$ be the sequence of associated coefficient functionals we need to show that each f_n is continuous on (E, τ). Let $x = \sum_{i=1}^{\infty} \lambda_i x_i \in E$. Then ,for each p_m,

$$p_m(x - \sum_{i=1}^{k} \lambda_i x_i) \to 0 \text{ as } k \to \infty.$$

Fix f_n. Then ,for any p_m,

$$p_m(f_n(x)x_n) = p_m(\sum_{i=1}^{n} f_i(x)x_i - \sum_{i=1}^{n-1} f_i(x)x_i)$$

or

$$\begin{aligned} |f_n(x)|\, p_m(x_n) &\leq p_m(\sum_{i=1}^{n} f_i(x)x_i) + p_m(\sum_{i=1}^{n-1} f_i(x)x_i) \\ &\leq p'_m(x) + p'_m(x) = 2p'_m(x) \end{aligned}$$

Since $x_n \neq 0$ and E is Hausdorff, there exists m_0 such that $p_{m_0}(x_n) \neq 0$. Then with $m = m_0$

$$|f_n(x)| \leq 2/(P_{m_0}(x_n))\acute{P}_n(x).$$

This implies that f_n is τ'–continuous on E. Since $\tau' = \tau$, f_n is τ-continuous on E. Thus $\{x_n\}$ is a Schauder basis on E. ■

Definition 1.5.9. Let E be TVS. A series $\sum_{k=1}^{\infty} x_k$ in E is called *unconditionally convergent* if, for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{k=1}^{\infty} x_{\sigma(k)}$ is convergent in E.

Clearly, each unconditional convergent series is convergent in the ordinary sense, but the converse is not true. For example, the series $\sum_{n=1}^{\infty} (-1)^n(1/n)$ is convergent but not conditionally convergent in $\mathbb{R}$ (since $\sum_{n=1}^{\infty} 1/n$ is not convergent).

Definition 1.5.10. A basis $\{x_n\}$ is a TVS E is called an *unconditional basis* if each series $x = \sum_{i=1}^{\infty} f_i(x)x_i \in E$ is unconditionally convergent.

Definition 1.5.11. A series $\sum_{n=1}^{\infty} y_n$ in a TVS E is called

(a) *boundedly-multiplier convergent* if, for each $\{\alpha_n\} \in \ell_\infty$, $\sum_{n=1}^{\infty} \alpha_n y_n$ is convergent in E.

(b) *subseries convergnt* if each subseries $\sum_{k=1}^{\infty} y_{n_k}$ of $\sum_{n=1}^{\infty} y_n$ is convergent in E.

(c) *unordered convergnt* if the net $\{ \sum_{k\in F}^{\infty} y_k : F \in \Im\}$ is convergent in E, where $\Im$ is the collection of all finite subsets of $\mathbb{N}$ ordered by inclusion.

We state the following theorems without proofs.

Theorem 1.5.12. *Let $\{x_n\}$ be an unconditional basis in a TVS E. Then, for each $x = \sum_{i=1}^{\infty} f_i(x)x_i$ and each $\{\alpha_n\} \in \ell_\infty$, the series $\sum_{i=1}^{\infty} \alpha_i f_i(x)x_i$ is convergent in E; i.e. $\ell_\infty \bigotimes A \subseteq A$.*

Theorem 1.5.13. (Equivalent conditions for unconditional bases) *Let E be a Fréchet space and $\{x_n\}$ a basis of E. Then the following statements are equivalent :*

(a) $\{x_n\}$is unconditional.

(b) For each $x \in E$, the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is unordered convergent.

(c) For each $x \in E$,the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is boundedly mutiplier convergent.

(d) For each $x \in E$,the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is subseries convergent.

(e) For each $x \in E$ and each sequence $\{\varepsilon_i\}$, $\varepsilon_i = \pm 1$, the series $\sum_{i=1}^{\infty} \varepsilon_i f_i(x)x_i$ is convergent.

1.6 Orthogonal Bases in Topological Algebras

Definition 1.6.1. Let A be a Hausdorff topological algebra. A basis $\{x_n\}$ for A is called :

(a) *semiorthogonal* if $x_i x_j = 0$ for $i \neq j$.

(b) *quasiorthogonal* if $x_i x_j = 0$ for $i \neq j$ and $x_i^2 \neq 0$.

(c) *orthogonal* if $x_i x_j = \delta_{ij} x_i = \{ {}^{0 \text{ if } i \neq j}_{x_i \text{ if } i=j}$;(that is , $x_i x_j = 0$ if $i \neq j$ and $x_i^2 = x_i$).

Lemma 1.6.2. *(i)* $(c) \Rightarrow (b) \Rightarrow (a)$.

(ii) If A has identiy e, then $(a) \Rightarrow (b)$.

Proof. (i) Cleary $(c) \Rightarrow (b) \Rightarrow (a)$.

(ii) Let $e = \sum_{i=1}^{\infty} \lambda_i x_i$ .Then , for any $i \geq 1, x_i = x_i e = x_i (\sum_{i=1}^{\infty} \lambda_i x_i) = \lambda_i x_i^2$. Since $x_i \neq 0$, we have $\lambda_i \neq 0$: hence $x_i^2 = x_i / \lambda_i \neq 0$. ■

Theorem 1.6.3. *Let A be a topological algebra with a basis $\{x_n\}$ and $\{f_n\}$ the associated sequence of coordinate functionals.*

(i) If $\{x_n\}$ is semiorthogonal , then $xy = yx = \sum_{i=1}^{\infty} f_i(x) f_i(y) x_i^2$ for all $x, y \in A$;hence A is commutative. If , in adition , A has identity e, then $x_i = f_i(e) x_i^2$ for all $i \geq 1$

(ii) If $\{x_n\}$is orthogonal , then $f_i(xy) = f_i(x) f_i(y)$ for all $i \geq 1$; hence each f_i is a m.l.f on A . If, in addition , A has identity e, then $f_i(e) = 1$ for all $i \geq 1$.

Proof. (i) Let $x = \sum_{i=1}^{\infty} f_i(x) x_i$, $y = \sum_{j=1}^{\infty} f_j(y) x_j \in A$. Then

$$\begin{aligned} xy &= (\sum_{i=1}^{\infty} f_i(x) x_i)(\sum_{j=1}^{\infty} f_j(x) x_j) = \lim_{n\to\infty} (\sum_{i=1}^{n} f_i(x) x_i) \lim_{m\to\infty} (\sum_{j=1}^{m} f_j(y) x_j) \\ &= \lim_{\substack{n\to\infty \\ m\to\infty}} (\sum_{i=1}^{n} f_i(x) x_i) (\sum_{j=1}^{m} f_j(y) x_j) = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{i=1}^{\min(n,m)} f_i(x) f_i(y) x_i^2 \\ &= \sum_{i=1}^{\infty} f_i(x) f_i(y) x_i^2 \end{aligned}$$

Similarly, $yx = \sum_{i=1}^{\infty} f_i(y) f_i(x) x_i^2 = \sum_{i=1}^{\infty} f_i(x) f_i(y) x_i^2$. Thus $xy = yx$, and so A is commutative.

Now , if A has also an identity e, then $e = \sum_{j=1}^{\infty} f_j(e) x_i = \lim_{n\to\infty} \sum_{j=1}^{n} f_j(e) x_j$. So , for any $i \geq 1$,

$$x_i = x_i e = x_i (\lim_{n\to\infty} \sum_{j=1}^{n} f_j(e) x_j) = \lim_{n\to\infty} x_i . \sum_{j=1}^{n} f_j(e) x_j = \lim_{n\to\infty} f_i(e) x_i^2 = f_i(e) x_i^2$$

(ii) Suppose $\{x_n\}$ is orthogonal $(i.e. x_i x_j = 0$ for $i \neq j$ and $x_i^2 = x_i)$. Hence, for any $x, y \in A$,

$$xy = \sum_{i=1}^{\infty} f_i(x) f_i(y) x_i^2 = \sum_{i=1}^{\infty} f_i(x) f_i(y) x_i.$$

But also $xy = \sum_{i=1}^{\infty} f_i(xy) x_i$. By uniqueness of representation of xy, $f_i(xy) = f_i(x) f_i(y)$ for all $i \geq 1$;that is, each f_i is multiplicative.

If A has also an identity e, then $x_i = f_i(e) x_i^2 = f_i(e) x_i$. By the uniqueness of representation of x_i, we have $f_i(e) = 1$ for all $i \geq 1$. ■

Theorem 1.6.4. *Let* $\{x_n\}$*and* $\{y_n\}$ *be orthogonal basis of a topological algebra A. Then* $\{x_n\} = \{y_n\}$.

Proof. (I) We first show that , for any $x_i \in \{x_n\}$, there is $y_i \in \{y_n\}$ such that $x_i y_j \neq 0$.[Suppose this is not true. Then $x_i y_i = 0$ for all $j \geq 1$. So if $x_i = \sum_{n=1}^{\infty} \lambda_n y_n$,

$$x_i^2 = x_i x_i = x_i(\sum_{n=1}^{\infty} \lambda_n y_n) = \sum_{n=1}^{\infty} \lambda_n x_i y_n = \sum_{n=1}^{\infty} \lambda_n .0 = 0,$$

which is impossible (since , by orthogonality , $x_i^2 = x_i \neq 0$).

(II) Using (I) , we now show that $x_i = y_i$:

(i) First, writing $x_i = \sum_{n=1}^{\infty} \lambda_n y_n$,

$$x_i y_j = (\sum_{n=1}^{\infty} \lambda_n y_n) y_i = \lambda_j y_j y_j = \lambda_j y_j^2 = \lambda_j y_j.$$

Multiplying on the left by x_i, we get $x_i^2 y_j = \lambda_j x_i y_j$, or $x_i y_j = \lambda_j x_i y_j$, or $(1 - \lambda_j) x_i y_j = 0$. Therfore $\lambda_j = 1$ (since $x_i y_j \neq 0$). Hence $x_i y_j = y_j$.

(ii) Next ,writing $y_j = \sum_{n=1}^{\infty} \mu_n x_n$,

$$x_i y_j = x_i(\sum_{n=1}^{\infty} \mu_n x_n) = \mu_i x_i x_i = \mu_i x_i^2 = \mu_i x_i.$$

Multiplying on the right by y_j, $x_i y_j^2 = \mu_i x_i y_j$, or $x_i y_j = \mu_i x_i y_j$, or $(1 - \mu_i) x_i y_j = 0$. Therfore $\mu_i = 1$ (since $x_i y_j \neq 0$). Hence $x_i y_j = x_i$. Thus $x_i = x_i y_j = y_j$.

Conversely, for any $y_j \in \{y_n\}$, there exists $x_j \in \{x_n\}$ such that $y_i x_j \neq 0$. By symmetry , the above argument implies that $y_i = y_i x_j = x_i$. Thus $\{x_n\} = \{y_n\}$.

Recall an algebra A is called *faithful* (or *proper* , or *without order*) if, for any $x \in A$, $xA = \{0\}$ or $Ax = \{0\}$ implies that $x = 0$. ■

Theorem 1.6.5. *Let A be topological algebra with a quasiorthogonal (in particular ,orthogonal) basis* $\{x_n\}$. *If, for* $x \in A$, $xx_i = 0$ *for all* $i \geq 1$, *then* $x = 0$. *Hence A is faithful.*

Proof. Let $x = \sum_{j=1}^{\infty} \lambda_j x_j \in A$. Suppose $xx_i = 0$ for all $i \geq 1$. Then $(\sum_{j=1}^{\infty} \lambda_j x_j)x_i = 0$, or $\lambda_i x_i x_i = 0$, $\lambda_i = 0$ for all $i \geq 1$ (since $x_i^2 = x_i \neq 0$). Hence $x = \sum \lambda_i x_i = 0$. ■

Definition 1.6.6. Let A be an algebra . An element $x \in A$ is called *nilpotent* if $x^n = 0$ for some $n \geq 1$.

Theorem 1.6.7. *Let A be a topological algebra with an orthogonal basis* $\{x_n\}$.*Then , for any* $x \in A$, $x \neq 0 \Leftrightarrow x^n \neq 0$ *for any* $n \geq 1$. *Hence A has no non-zero nilpotent element .*

Proof. Let $x = \sum_{i=1}^{\infty} \lambda_i x_i \in A$. Then , by a result, $x^2 = \sum_{i=1}^{\infty} \lambda_i^2 x_i^2 = \sum_{i=1}^{\infty} \lambda_i^2 x_i$ and by induction $x^n = \sum_{i=1}^{\infty} \lambda_i^n x_i$. Since $\lambda_i \neq 0 \Leftrightarrow \lambda_i^n \neq 0$ for any $n \geq 1$, we have $x \neq 0 \Leftrightarrow x^n \neq 0$ for any $n \geq 1$.Hence A has no non-zero nilpotent element. ■

Note. If A is a Fréchet (in particular Banach) algabra, then, using the open mapping theorem, we have earlier shown that each basis in A is a Schauder basis. We now show that a quasi-orthogonal basis in any topological algebra A is a Schauder basis without using the open mapping theorem.

Theorem 1.6.8. *Every quasi-orthogonal basis* $\{x_n\}$*in a Hausdorff topological algebra A is a Schauder basis .*

Proof. Let $\{f_n\}$ be the associated sequence of coordinate functionals for the basis $\{x_n\}$.To show that each f_n is continuous,fix f_i and let $\varepsilon > 0$. By orthogonality of $\{x_n\}$, for any $x \in A$,

$$xx_i = (\sum_{j=1}^{\infty} f_j(x)x_j)x = f_i(x)x_i^2 \text{ with } x_i^2 \neq 0$$

Since A is Hausdorff and $x_i^2 \neq 0$, there exists a balanced neighbourhood U of 0 in A such that $x_i^2 \notin U$. Since multiplication in A is jointly continuous, there is a abalanced neighbourhood V of 0 in A such that $V^2 \subseteq U$. Then, since $x_i^2 \notin U$, we have $x_i \notin V$ (because if $x_i \in V$, then $x_i^2 \in V^2 \subseteq U$, a contradiction).

Let $r_0 = \inf\{r > 0 : x_i^2 \in rU\}$. Since $x_i^2 \notin U$, $r_0 \geq 1$. Since V is absorbing , there exists $t_0 > 0$ such that $x_i \in t_0V$. Put $W = (\varepsilon r_0/t_0)V$. Clearly W is a neighbourhood of 0 in A.We now show that $|f_i(x)| \leq \varepsilon$ for all $x \in W$. Let $x \in W$. Then , as in (1) , $xx_i = f_i(x)x_i^2$ and so

$$f_i(x)x_i^2 = xx_i \in ((\varepsilon r_0 V)/t_0)t_0V = \varepsilon r_0 V^2 \subseteq \varepsilon r_0 U$$

Supppose $f_i(x) \neq 0$.Then, since U is balanced, by above

$$x_i^2 \in (1/f_i(x))\varepsilon r_0 U = (1/\,|f_i(x)|)\varepsilon r_0 U.$$

Since $r_0 = \inf\{r > 0 : x_i^2 \in rU\}$, the above implies that $r_0 \leq (1/\,|f_i(x)|)\varepsilon r_0$, or $|f_i(x)| \leq \varepsilon$. If $f_i(x) = 0$, clearly $|f_i(x)| = 0 < \varepsilon$.Thus $|f_i(x)| \leq \varepsilon$ for all $x \in W$.

This proves that f_i is continuous on A, and therefore $\{x_n\}$ is a Schauder basis in A. ■

Note. The above theorem is also true for Hausdorff semi-topological algebras: replace $V^2 \subseteq U$ by $x_iV \subseteq U$ in the above proof.

Let A be a TA with orthogonal basis $\{x_n\}$. Recall that, for any $B \subseteq \{x_n\}$, the linear span of B is given by

$$span\ (B) = \{\sum_{k=1}^{m} \alpha_k x_k : \alpha_k \in \mathbb{K}, x_k \in B, m \geq 1\}.$$

and $rad\ (A) = \cap\{M : M$ a maximal modular ideal of $A\} = \cap\{f^{-1}(0) : f \in \triangle(A)\}$

Lemma 1.6.9. *Let* $x = \sum_{i=1}^{\infty} \lambda_i x_i \in A$. *Then* $x \in \overline{span\ (B)} \Leftrightarrow \lambda_i = f_i(x) = 0$ *whenever* $x_i \notin B$.

Proof. ($\Rightarrow$) Let $x \in \overline{span\ (B)}$, but suppose that $\lambda_i \neq 0$ for some $x_i \notin B$. choose a net $\{y\alpha\} \subseteq span\ (B)$ such that $y\alpha \to x$. Then , by continuity of coefficient functionals f_j, $f_i(y\alpha) \to f_i(x) = \lambda_i \neq 0$, a contradiction.

($\Leftarrow$) Let $x = \sum_{i=1}^{\infty} \lambda_i x_i \in B$ such that $\lambda_i = 0$ whenever $x_i \notin B$. Then $S_n = \sum_{i=1}^{n} \lambda_i x_i \in span\ (B)$ for all $n \geq 1$. Hence $x = \sum_{i=1}^{\infty} \lambda_i x_i = \lim_{n \to \infty} S_n \in \overline{span\ (B)}$. ■

Lemma 1.6.10. *The following statements are equivalent :*

(a) $x_i \in rad\,(A)$.

(b) $x_i^2 \in \overline{span\,(B)}$, where $B = \{x_j : x_i^2 = 0\}$.

(c) $f_i(x_i^2) = 0$.

(d) $x_i^3 = 0$.

Proof. First we note that $f_j(x_i^2)x_j^2 = 0$ for all $j \neq i$ as follows:

$$x_i^2 = x_i x_i = (\sum_{j=1}^{\infty} f_j(x_i)x_j)(\sum_{k=1}^{\infty} f_k(x_i)x_k) = \sum_{k=1}^{\infty} f_k(x_i^2)x_k^2 = \sum_{k=1}^{\infty} f_k(x_i^2)x_k$$

and so, for all $j \neq i$, $x_j x_i^2 = (x_j x_i)x_i = 0.x_i = 0$,or $x_j(\sum_{k=1}^{\infty} f_k(x_i^2)x_k) = 0$, or $f_j(x_i^2)x_j^2 = 0$.

$(a) \Rightarrow (b)$ Suppose $x_i \in rad\,(A)$, but $x_i^2 \notin \overline{span\,(B)}$, in which $B = \{x_j : x_j^2 = 0\}$. But for some j, $f_j(x_i^2) \neq 0$ and $x_j^2 \neq 0$. But then we have $j = i$ and so $\lambda_i = f_i(x_i^2) \neq 0$. Now define a sequance $\{y_k\}$ as follows: $y_k = \{ {}^{x_k \text{ for } k \neq i}_{x_i/\lambda_i \text{ for } k=i}$. Then $\{y_k\}$ is a basis of A and

$$y_i^2 = x_i^2/\lambda_i^2 = 1/\lambda_i^2 \sum_{j=1}^{\infty} f_j(x_i^2)x_j = \sum_{j=1}^{\infty} (f_j(x_i^2)/\lambda_i^2)x_j,$$

and hence $f_i(y_i)^2 = 1$ and $f_j(y_i^2) = 0$ for $j \neq i$. Thus, for any $x, y \in A$, $xy = \sum_{k=1}^{\infty} f_k(x)f_k(y)y_k^2$ and so

$$f_i(xy) = f_i(\sum_{k=1}^{\infty} f_i(x)f_i(y)y_k^2) = \sum_{k=1}^{\infty} f_i(x)f_i(y)f_i(y_k^2) = f_i(x)f_i(y)f_i(x_k^2)$$

and $f_i(y_k) = 1 \neq 0$ for all $k \geq 1$. Thus $y_i \notin rad\,(A)$, or $x_i/\lambda_i \notin rad\,(A)$, or $x_i \notin rad\,(A)$,a contradiction (since $x_i \in rad\,(A)$ means $f(x_i) = 0$ for all $f \in \Delta(A)$ and in particular $f_i(x_i) = 0$).

$(b) \Rightarrow (c)$ Suppose $x_i^2 \in \overline{span\,(B)}$, where $B = \{x_j : x_j^2 = 0\}$ for $j \neq i, 0 = x_j x_i^2 = x_j(\sum_{k=1}^{\infty} f_k(x_i^2)x_k) = f_j(x_i^2)x_j^2$. Therefore $f_j(x_i^2) = 0$ whenever $x_j^2 \neq 0$. Since, by hypothesis, $x_i^2 \in \overline{span\,(B)}$,the above implies that $f_i(x_i^2) = 0$.

$(c) \Rightarrow (d)$. Suppose $f_i(x_i^2) = 0$. Since $x_i^2 = \sum_{j=1}^{\infty} f_j(x_i^2)x_j$, we have

$$x_i^3 = x_i x_i^2 = x_i(\sum_{j=1}^{\infty} f_j(x_i^2)x_j = f_i(x_i^2)x_i^2 = 0.x_i^2 = 0.$$

$(d) \Rightarrow (a)$. Suppose $x_i^3 = 0$. Then, for any $f \in \Delta(A)$,

$$(f(x_i))^3 = f(x_i)f(x_i)f(x_i) = f(x_i^3) = f(0) = 0$$

and so, $f(x_i) = 0$. Thus $x_i \in \cap\{f^{-1}(0) : f \in \Delta(A)\} = rad\,(A)$. ■

Remark. As seen above, each semiorthogonal basis $\{x_n\}$ in a TA satisfies: $x_n^3 = 0$, or $x_n^3 = \alpha x_n^2$ for $\alpha \neq 0$. Now by a suitable change of transformation, we may change the given semiorthogonal basis $\{x_n\}$ to a new semiorthogonal basis $\{y_m\}$ with $y_m^3 = y_m^2$. Thus the corrsponding coefficient functionals f_m are continuous.

Theorem1.6.11. *$rad\ (A) = \overline{span\ (B)}$, where B is the subset of $\{x_n\}$ satisfying any one of the four conditions (a)-(d) of Lemma 1.6.10*

Proof. Since all the conditions of Lemma 1.6.10 are equivalent, we prove the theorem under (d). Let $B = \{x_i : x_i^3 = 0\}$.

($\Rightarrow$) If $x \in \overline{span\ (B)}$, with $x = \sum_{i=1}^{\infty} \lambda_i x_i$, then, by Lemma 1.6.9,

$$\lambda_i \neq 0 \Leftrightarrow \times_i \in B \Leftrightarrow x_i^3 = 0.$$

$$x^3 = \left[\sum_{i=1}^{\infty} \lambda_i x_i\right]^3 = \sum_{i=1}^{\infty} \lambda_i^3 x_i^3 = 0.$$

This implies that , for any $f \in \triangle(A)$,

$$[f(x)]^3 = f(x).f(x).f(x) = f(x^3) = f(0) = 0,$$

and so $f(x) = 0$. Hence $x \in rad\ (A)$. So $\overline{span\ (B)}, \subseteq rad(\ A)$.

($\Leftarrow$) Suppose $x \in rad\ (A)$, but $x \notin \overline{span\ (B)}$. Then by Lemma 1.6.10, there exists some j such that $\lambda_j \neq 0$ and $x_j \in B$, or $x_j^3 = 0$. Then choosing a suitable basis as in Lemma 1.6.11 $(a) \Rightarrow (b)$, it follows that $x \notin rad\ (A)$, a contradiction. Hence $x \in \overline{span\ (B)}$, and so $rad\ (A) \subseteq \overline{span\ (B)}$.

Thus $rad\ (A) = \overline{span(\ B)}$. ■

Corollary 1.6.12. *A TA A with a Schauder orthogonal basis (in particualr, a Hausdorff TA with an orthogonal basis) is semisimple .*

Proof. Let $\{x_n\}$ be an orthogonal basis with $\{f_n\}$ the sequence of coefficient functionals. Let $x = \sum_{i=1}^{\infty} f_i(x) x_i \in rad\ (A)$. Then $f(x) = 0$, for all $f \in \triangle(A)$, $f_i(x) = 0$, for all $i \geq 1$ and so

$$x = \sum_{i=1}^{\infty} f_i(x) x_i = \sum_{i=1}^{\infty} 0.x_i = 0$$

Hence $rad\ (A) = \{0\}$, and so A is semisimple. ■

Theorem 1.6.13. *Let A be a topological algebra with a Schauder orthogonal basis $\{x_n\}$ and $\{f_n\}$ the associated coeffecient functionals.*

(a) For each $i \geq 1$, $M_i = \{x \in A : f_i(x) = 0\}$ is a closed maximal ideal in A. Hence $\Delta(A) = \{M_n\} = \{f_n\}$.

(b) Each proper closed ideal of A is contained in some M_i.

(c) The set $\{f_n\}$ with the relative weak* topology $\sigma(A^/, A)$ (i.e. the Gelfand topology) is homeomorphic with the discrete countable space $\mathbb{N}$ of natural numbers.

Proof. (a) Since each f_i is a continuous multiplicative linear functional on A, it is easy to verify that M_i is a closed maximal modular ideal

(b) Let I be a proper closed ideal in A, but suppose that $I \nsubseteq M_i$ for all $i \geq 1$. Then there is $x \in I$ such that $f_i(x) \neq 0$ for all $i \geq 1$. Since $xx_i = (\sum_{j=1}^{\infty} f_j(x))x_i = f_i(x)x_i$, we have $x_i = (xx_i/f_i(x)) \in I$ for all $i \geq 1$ (since $x \in I$ and I is an ideal).

Now , for any $y \in A$, $\sum_{i=1}^{k} f_i(y)x_i \in I$ and so

$$y = \sum_{i=1}^{\infty} f_i(y)x_i = \lim_{k\to\infty} \sum_{i=1}^{k} f_i(y)x_i \in \bar{I}.$$

This implies that $\bar{I} = A$ and so I is dense in A. Since I is closed, $I = \bar{I} = A$. This is a contradiction since I is proper .

(c) Clearly, a subbasis at any f_i for the Gelfand topologiy τ_G on $\{f_n\}$ consists of all sets of the form:

$$V_i = \{f_j : |f_j(x_i) - f_i(x_i)| \leq 1\}.$$

Since $f_i(x_i) = 1$ for all $i \geq 1$ and $f_j(x_i) = 0$ for $j \neq i$, we have $V_i = \{f_i\}$, a singleton. Hence $(\{f_n\}, \tau_G)$ is a discrete space. Since it is countable, it is homeomorphic to the discrete space $\mathbb{N}$ under the homeomorphism $\phi : \{f_n\} \to \mathbb{N}$ given by $\phi(f_k) = k$ for all $k \geq 1$. ■

CHAPTER 2

MULTIPLIERS ON BANACH ALGEBRAS

In this chapter, we study the notions of left multipliers, right multipliers, and multipliers on both algebras and Banach algebras. If A is a faithful commutative algebra, then it is shown that $M_l(A) = M_r(A) = M(A)$; if, in addition, A has an identity, then $M(A)$ is algebraically isomorphic with the algebra $\{L_x : x \in A\}$ of regular representations.We also consider the problems of embedding of A into the algebras of multipliers $M_l(A), M_r(A)$ and $M(A)$ as (left, right) ideal. If A is a Banach algebra, then under certain conditions (such as A is faithful or has an approximate identity), we establish the linearity and continuity of the multipliers $T : A \to A$. Finally, we introduce uniform and strict topologies on the multiplier algebras and study their completeness and dense properties. These results are due to Johnson [15, 16], Busby [5] and Wang [26].

2.1 Basic Properties of Multipliers on Algebras

In this section, we present some definitions and results of Johnson [15] about multipliers on algebras.

Definition 2.1.1. Let A be an algebra. A mapping $T : A \to A$ is called a

(i) *left multiplier* on A if

$$T(ab) = T(a)b \quad \text{for all } a, b \in A;$$

(ii) *right multiplier* on A if

$$T(ab) = aT(b) \quad \text{for all } a, b \in A;$$

(iii) *multiplier* on A if

$$aT(b) = T(a)b \quad \text{for all } a, b \in A.$$

Definition 2.1.2. Let A be an algebra. Then

(1) A is said to be *left faithful* if, for any $x \in A$, $xA = \{0\}$ implies that $x = 0$.

(2) A is said to be *right faithful* if, for any $x \in A$, $Ax = \{0\}$ implies that $x = 0$.

(3) A is said to be *faithful* if it is both left and right faithful.

Notations 2.1.3. For any algebra A, let

$$M_l(A) = \text{the set of all left multipliers } T : A \to A,$$

$$M_r(A) = \text{the set of all right multipliers } T : A \to A,$$

$$M(A) = \text{the set of all multipliers } T : A \to A.$$

Definition 2.1.4. Let A be an algebra. For any $x \in A$, define $L_x, R_x : A \to A$ by

$$\begin{aligned} L_x(a) &= xa \quad \text{for all } a \in A, \\ R_x(a) &= ax \quad \text{for all } a \in A. \end{aligned}$$

The following theorem provides us with examples of above defined multipliers.

Theorem 2.1.5. *Let A be any algebra, and let $x \in A$. Then:*

(a) $L_x \in M_l(A)$ and $R_x \in M_r(A)$.

(b) Suppose A is commutative. Then $L_x, R_x \in M(A)$ and also $L_x = R_x$.

Proof. (a) For any $a, b \in A$,

$$\begin{aligned} L_x(ab) &= x(ab) \\ &= (xa)b = L_x(a)b, \end{aligned}$$

$$\begin{aligned} R_x(ab) &= (ab)x \\ &= a(bx) = aR_x(b). \end{aligned}$$

Hence $L_x \in M_l(A)$ and $R_x \in M_r(A)$.

(b) Suppose A is commutative. Then for any $a, b \in A$, we have

$$\begin{aligned} aL_x(b) &= a(xb) = (ax)b \\ &= (xa)b = L_x(a)b, \end{aligned}$$

$$\begin{aligned} aR_x(b) &= a(bx) = a(xb) \\ &= (ax)b = R_x(a)b. \end{aligned}$$

Further, for any $a \in A$,

$$L_x(a) = xa = ax = R_x(a).$$

Hence $L_x, R_x \in M(A)$ and $L_x = R_x$. ■

Theorem 2.1.6. *Let A be a commutative algebra. Then $T \in M_l(A)$ if and only if $T \in M_r(A)$; hence $M_l(A) = M_r(A)$.*

Proof. Let $T \in M_l(A)$. Then, for any $a, b \in A$,

$$T(ab) = T(ba) = T(b)a = aT(b)$$

Hence $T \in M_r(A)$. Similarly, if $T \in M_r(A)$, then $T \in M_l(A)$. ■

Theorem 2.1.7. *Let A be an algebra. Then*

(a) $M_l(A) \cap M_r(A) \subseteq M(A)$.

(b) If A is faithful, then $M(A) \subseteq M_l(A) \cap M_r(A)$ and hence $M(A) = M_l(A) \cap M_r(A)$.

(c) If A is commutative and faithful, then $M_l(A) = M_r(A) = M(A)$.

Proof. (a) Let $T \in M_l(A) \cap M_r(A)$. Then, for any $a, b \in A$,

$$aT(b) = T(ab) = T(a)b$$

Hence $T \in M(A)$, and so $M_l(A) \cap M_r(A) \subseteq M(A)$.

(b) Suppose that A is faithful, and let $T \in M(A)$ and $a, b \in A$. Then, for any $c \in A$,

$$cT(ab) = T(c)(ab) \quad (\because T \in M(A))$$

$$\begin{aligned} &= (T(c)a)b \\ &= (cT(a))b \quad (\because T \in M(A)) \\ &= c(T(a)b). \end{aligned}$$

So $c(T(ab) - T(a)b) = 0$ for all $c \in A$. Since A is left faithful, we have $T(ab) = T(a)b$, and so, $T \in M_l(A)$. Since A is right faithful, $T \in M_r(A)$, as follows

$$\begin{aligned} T(ab)c &= (ab)T(c) = aT(b)c = T(a)bc \\ &\implies T(ab) = T(a)b. \end{aligned}$$

Thus $M(A) \subseteq M_l(A) \cap M_r(A)$.

Consequently, by part (a), $M(A) = M_l(A) \cap M_r(A)$.

(c) By Theorem 2.1.6 , $M_l(A) = M_r(A)$. So, by part (b), $M(A) = M_l(A) = M_r(A)$. ■

We now discuss the linearity of multipliers as follows.

Definition 2.1.8. Let A and B two algebras and $T : A \to B$ a mapping.

(i) T is called *linear* if

(a) T is *additive*, i.e. $T(x + y) = T(x) + T(y)$ for all $x, y \in A$,

(b) T is *homogeneous, i.e.* if $T(\alpha a) = \alpha T(a)$ *for all* $\alpha \in \mathbb{K}$ *and* $a \in A$.

(ii) T is called an *algebra homomorphism* if T is linear and

$$T(xy) = T(x)T(y) \text{ for all } x, y \in A..$$

Theorem 2.1.9. *Let A be an algebra.*

(a) Suppose A is left faithful. If $T \in M_l(A)$, then T is homogeneous.

(b) Suppose A is right faithful. If $T \in M_r(A)$, then T is homogeneous.

(c) Suppose A is faithful. If $T \in M(A)$, then T is linear.

Proof. (a) Let $T \in M_l(A)$, and let $a \in A$ and $\alpha \in \mathbb{K}$. Then, for any $c \in A$,

$$T(\alpha a)c = T(\alpha a.c) = T(a.\alpha c)$$

$$= \quad T(a).\alpha c = \alpha T(a).c$$

Since A is left faithful, $T(\alpha a) = \alpha T(a)$.

(b) Let $T \in M_r(A)$, and let $a \in A$ and $\alpha \in \mathbb{K}$. Then, for any $c \in A$,

$$\begin{aligned} c.T(\alpha a) &= T(c\alpha a) = T(\alpha c a) \\ &= \alpha c T(a) = c\alpha T(a). \end{aligned}$$

Since A is right faithful, $T(\alpha a) = \alpha T(a)$.

(c) Let $T \in M(A)$, and let $a, b \in A$ and $\alpha, \beta \in \mathbb{K}$. Then, for any $c \in A$,

$$\begin{aligned} cT(\alpha a + \beta b) &= T(c)(\alpha a + \beta b) = \alpha T(c)a + \beta T(c)b \\ &= \alpha c T(a) + \beta c T(b) = c(\alpha T(a) + \beta T(b)). \end{aligned}$$

Since A is faithful, $T(\alpha a + \beta b) = \alpha T(a) + \beta T(b)$; that is, T is linear. ■

Theorem 2.1.10. *Let A be an algebra.*

(a) If $T \in M_l(A)$ and T^{-1} exists, then $T^{-1} \in M_l(A)$.

(b) If $T \in M_r(A)$ and T^{-1} exists, then $T^{-1} \in M_r(A)$.

(c) If $T \in M(A)$ and T^{-1} exists, then $T^{-1} \in M(A)$.

Proof. (a) For any $a, b \in A$,

$$\begin{aligned} T^{-1}(ab) &= T^{-1}[I(a).b] = T^{-1}[(TT^{-1})(a).b] = T^{-1}[T(T^{-1}a).b] \\ &= T^{-1}[T\{(T^{-1}a).b\}] = (T^{-1}T)[(T^{-1}a).b] \\ &= I[(T^{-1}a).b] = T^{-1}(a).b. \end{aligned}$$

(b) This is similar to part (a).

(c) For any $a, b \in A$,

$$\begin{aligned} T^{-1}(a).b &= T^{-1}(a).(TT^{-1})(b) = T^{-1}(a).T(T^{-1}b) \\ &= T(T^{-1}a).(T^{-1}b) = a.T^{-1}(b). \end{aligned}$$

2.2. The Multiplier Algebras $M_l(A)$, $M_r(A)$ and $M(A)$.

In this section, we study algebraic versions of some results of Wang [26], Busby [5] and Johnson [15].

Let A be an algebra. For any $T_1, T_2 : A \to A$ and $\alpha \in \mathbb{K}$, we define operations $T_1+T_2, \alpha T_1, T_1T_2$ by

$$\begin{aligned}(T_1+T_2)(x) &= T_1(x)+T_2(x),\\ (\alpha T_1)(x) &= \alpha T_1(x),\\ (T_1T_2)(x) &= T_1(T_2(x)), \qquad x \in A.\end{aligned}$$

The identity mapping $I : A \to A$ is given by $I(x) = x$, $x \in A$.

Theorem 2.2.1. *Let A be an algebra. Then*

(a) $M_l(A)$ is an algebra with identity I.

(b) $M_r(A)$ is an algebra with identity I.

Proof. (a) To show that $M_l(A)$ is a vector space , let $T_1, T_2 \in M_l(A)$ and $\alpha, \beta \in \mathbb{K}$. Then , for any $x, y \in A$,

$$\begin{aligned}(\alpha T_1+\beta T_2)(xy) &= \alpha T_1(xy)+\beta T_2(xy)\\ &= \alpha T_1(x)y+\beta T_2(x).y\\ &= (\alpha T_1(x)+\beta T_2(x).y\\ &= (\alpha T_1+\beta T_2)(x).y;\end{aligned}$$

hence $\alpha T_1 + \beta T_2 \in M_l(A)$. Therefore $M_l(A)$ is a vector space.To show that $M_l(A)$ is an algebra, let $T_1, T_2 \in M_l(A)$..Then, for any $x, y \in A$,

$$\begin{aligned}(T_1T_2)(xy) &= T_1[T_2(xy)]\\ &= T_1[T_2(x).y]\\ &= T_1(T_2(x)).y\end{aligned}$$

$$= \quad (T_1T_2)(x).y,$$

and so $T_1T_2 \in M_l(A)$. Hence $M_l(A)$ is an algerbra. If $I : A \to A$ is the identity map, then for any $x, y \in A$, $I(xy) = xy = I(x)y$, we have $I \in M_l(A)$. Thus $M_l(A)$ is an algebra with identity I.

(b) It is identical to that of part (a). ■

Theorem 2.2.2. *Let A be an algebra. Then:*

(i) $M(A)$ is a vector space.

(ii) If A is faithful, then $M(A)$ is a commutative algebra with identity I.

Proof. (i)To show that $M(A)$ is a vector space , let $T_1, T_2 \in M(A)$ and $\alpha, \beta \in \mathbb{K}$. Then , for any $x, y \in A$,

$$\begin{aligned} x(\alpha T_1 + \beta T_2)(y) &= \alpha x T_1(y) + \beta x T_2(y) \\ &= \alpha T_1(x)y + \beta T_2(x)y \\ &= (\alpha T_1(x) + \beta T_2(x))y \\ &= (\alpha T_1 + \beta T_2)(x)y; \end{aligned}$$

hence $\alpha T_1 + \beta T_2 \in M(A)$. Therefore $M(A)$ is a vector space.

(ii) Suppose A is faithful. To show that $M(A)$ is an algebra, let $T_1, T_2 \in M(A)$. Since A is faithful, $T_1, T_2 \in M_\ell(A) \cap M_r(A)$. Therefore, for any $x, y \in A$,

$$\begin{aligned} x(T_1T_2)(y) &= xT_1(T_2(y)) \\ &= T_1[x(T_2(y))] \\ &= T_1[T_2(x).y] \\ &= [T_1(T_2(x)].y \\ &= (T_1T_2)(x).y, \end{aligned}$$

and so $T_1T_2 \in M(A)$. Hence $M(A)$ is an algerbra.

To show that $M(A)$ is commutative , let $T_1, T_2 \in M(A)$, and $x \in A$. Then, for any $z \in A$,

$$z(T_1T_2)(x) \quad = \quad (T_1T_2)(z)x$$

$$
\begin{aligned}
&= T_1(T_2(z))x \\
&= T_2(z)T_1(x) \\
&= z(T_2T_1(x)) \\
&= z(T_2T_1)(x)
\end{aligned}
$$

Since A is faithful, $(T_1T)(x) = (TT_1)(x)$, $x \in A$. Hence $T_1T_2 = T_2T_1$, and so $M(A)$ is commutative. If $I : A \to A$ is the identity map, then for any $x, y \in A$, $xI(y) = xy = I(x)y$, we have $I \in M(A)$. Thus $M(A)$ is a commutative algebra with identity I. ■

Recall that, for any $x \in A$, the maps $L_x, R_x : A \to A$ given by $L_x(a) = xa, R_x(a) = ax \quad (a \in A)$ are such that $L_x \in M_l(A)$ and $R_x \in M_r(A)$.

Theorem 2.2.3. *Let A be an algebra, and let $x, y \in A$ and $\alpha \in \mathbb{K}$. Then*

(a) (i) $L_x + L_y = L_{x+y}$

(ii) $\alpha L_x = L_{\alpha x}$

(iii) $L_x L_y = L_{xy}$;

hence $\{L_x : x \in A\}$ *is a subalgebra of* $M_l(A)$.

(b) (i) $R_x + R_y = R_{x+y}$

(ii) $\alpha R_x + R_{\alpha x}$

(iii) $R_x R_y = R_{yx}$;

hence $\{R_x : x \in A\}$ *is a subalgebra of* $M_r(A)$.

Proof. (a) (i) For any $a \in A$,

$$
\begin{aligned}
(L_x + L_y)(a) &= L_x(a) + L_y(a) = xa + ya \\
&= (x + y)a = L_{x+y}(a);
\end{aligned}
$$

hence $L_x + L_y = L_{x+y}$.

(ii) For any $a \in A$,

$$\begin{aligned}\alpha L_x(a) &= \alpha(L_x(a)) = \alpha(xa)\\ &= (\alpha x)a = L_{\alpha x}(a);\end{aligned}$$

hence $\alpha L_x = L_{\alpha x}$.

(iii) For any $a \in A$,

$$\begin{aligned}(L_x L_y)(a) &= L_x(L_y(a)) = L_x(ya)\\ &= x(ya) = (xy)a = L_{xy}(a);\end{aligned}$$

hence $L_x L_y = L_{xy}$.

Clearly, by (i)-(iii), if $x, y \in A$ and $\alpha \in \mathbb{K}$,

$$L_x + L_y,\ \alpha L_x,\ L_x L_y \in \{L_z : z \in A\} \subseteq M_l(A).$$

Hence $\{L_x : x \in A\}$ is a subalgebra of $M_l(A)$.

(b) The proof is similar to part (a). ■

Theorem 2.2.4.. *Let A be an algebra.*

(a) For any $a \in A$ and $T \in M_l(A)$,

$$TL_a = L_{T(a)};$$

hence $\{L_x : x \in A\}$ is a left ideal in $M_l(A)$.

(b) For any $a \in A$ and $T \in M_r(A)$,

$$TR_a = R_{T(a)};$$

hence $\{R_x : x \in A\}$ is a left ideal in $Mr(A)$.

(c) For any $a \in A$ and $T \in M(A)$,

$$L_a T = L_{T(a)}, R_a T = R_{T(a)};$$

if A is faithful,

$$TL_a = L_{T(a)}, TR_a = R_{T(a)}.$$

(Here note that $\{L_x : x \in A\} \nsubseteq M(A)$ and $\{R_x : x \in A\} \nsubseteq M(A)$ unless A is commutative.)

Proof. (a) Let $a \in A$ and $T \in M_l(A)$. For any $c \in A$,

$$(TL_a)(c) = T(ac) = T(a)c = L_{T(a)}(c).$$

Hence $TL_a = L_{T(a)}$ and so $\{L_x : x \in A\}$ is a left ideal in $M_l(A)$

(b) Let $a \in A$ and $T \in M_r(A)$. For any $c \in A$,

$$(TR_a)(c) = T(ca) = cT(a) = R_{T(a)}(c).$$

Hence $TR_a = R_{T(a)}$ and so $\{R_x : x \in A\}$ is a left ideal in $Mr(A)$.

(c) Let $a \in A$ and $T \in M(A)$. For any $c \in A$,

$$(L_aT)(c) = aT(c) = T(a)c = L_{T(a)}(c);$$

$$(R_aT)(c) = T(c)a = cT(a) = R_{T(a)}(c).$$

Hence $L_aT = L_{T(a)}$, $R_aT = R_{T(a)}$.

If A is faithful, then $M(A) = M_l(A) \cap M_r(A)$. So by (a) and (b),

$$TL_a = L_{T(a)},\ TR_a = R_{T(a)}.$$

■

To embed A into $M_l(A)$ and $M_r(A)$, we need the following definition.

Definition 2.2.5. Let A be a algebra. We define mappings

(i) $\mu_l : A \to M_l(A)$ by

$$\mu_l(x) = L_x,\ \ x \in A;$$

(ii) $\mu_r : A \to M_r(A)$ by

$$\mu_r(x) = R_x,\ \ x \in A;$$

Note that

$$\mu_l(A) = \{L_x : x \in A\} \subseteq M_l(A);$$

$$\mu_r(A) = \{R_x : x \in A\} \subseteq M_r(A).$$

We can restate Theorem 2.2.4 as follows.

Theorem 2.2.6. *Let A be an algebra, and let μ_l, μ_r be the mappings as defined above. Then*

(a) $\mu_l(A)$ is a left ideal in $M_l(A)$.

(b) $\mu_r(A)$ is a left ideal in $M_r(A)$.

(c) If A is commutative and faithful, then $\mu_l(A)$ $(= \mu_r(A))$ is a 2-sided ideal in $M(A)$.

. We next study some properties of the embedding maps μ_l and μ_r.

Theorem 2.2.7. *Let A be an algebra, and let $\mu_l : A \to M_l(A)$ and $\mu_r : A \to M_r(A)$ be the mappings defined as before. Then*

(a) μ_l and μ_r are linear.

(b) (i) μ_l is an algebra homomorphism.

(ii) μ_r is an algebra anti-homomorphism.

(c) (i) μ_l is 1-1 $\Longleftrightarrow$ A is left faithful.

(ii) μ_r is 1-1 $\Longleftrightarrow$ A is right faithful.

(d) (i) μ_l is onto $\Longleftrightarrow$ A has a left identity.

(ii) μ_r is onto $\Longleftrightarrow$ A has a right identity.

Proof. (a) For any $x, y \in A$ and $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} \mu_l(\alpha x + \beta y) &= L_{\alpha x + \beta y} = L_{\alpha x} + L_{\beta y} \\ &= \alpha L_x + \beta L_y = \alpha \mu_l(x) + \beta \mu_l(y); \end{aligned}$$

$$\begin{aligned} \mu_r(\alpha x + \beta y) &= R_{\alpha x + \beta y} = R_{\alpha x} + R_{\beta y} \\ &= \alpha R_x + \beta R_y = \alpha \mu_r(x) + \beta \mu_r(y). \end{aligned}$$

Hence μ_l and μ_r are linear.

(b) (i) For any $x, y \in A$,

$$\mu_l(xy) = L_{xy} = L_x L_y = \mu_l(x)\mu_l(y).$$

Hence μ_l is an algebra homomorphism.

(ii) For any $x, y \in A$,

$$\mu_r(xy) = R_{xy} = R_y R_x = \mu_r(y)\mu_r(x).$$

So μ_r is an algebra anti-homomorphism.

(c) (i) μ_l is 1-1 $\Longleftrightarrow \mu_l(x) = 0 \Longrightarrow x = 0$

$\Longleftrightarrow L_x = 0 \Longrightarrow x = 0$

$\Longleftrightarrow L_x(a) = 0$ for all $a \in A \Longrightarrow x = 0$

$\Longleftrightarrow xa = 0$ for all $a \in A \Longrightarrow x = 0$

$\Longleftrightarrow xA = \{0\} \Longrightarrow x = 0$

$\Longleftrightarrow A$ is left faithful in X.

(ii) μ_r is 1-1 $\Longleftrightarrow \mu_r(x) = 0 \Longrightarrow x = 0$

$\Longleftrightarrow R_x = 0 \Longrightarrow x = 0.$

$\Longleftrightarrow R_x(a) = 0$ for all $a \in A \Longrightarrow x = 0$

$\Longleftrightarrow ax = 0$ for all $a \in A \Longrightarrow x = 0$

$\Longleftrightarrow Ax = \{0\} \Longrightarrow x = 0.$

$\Longleftrightarrow A$ is right faithful.

(d) (i) Suppose that μ_l is onto. Then $\mu_l(A) = M_l(A)$ and for each $T \in M_l(A)$ there exists some $a \in A$ such that $T = L_a$. Hence $I = L_e$ for some $e \in A$, since $I \in M_l(A)$. Thus $I(a) = L_e(a)$ for all $a \in A$ and $a = ea$ for all $a \in A$.Therefore, A has a left identity e.

Conversely, suppose A has a left identity e. To show that μ_l is onto, let $T \in M_l(A)$. Then , for any $a \in A$,

$$T(a) = T(ea) = T(e)a = L_{T(e)}(a),$$

that is, $T = L_{T(e)} = \mu_l(T(e)) \in \mu_l(A)$. Hence $M_l(A) \subseteq \mu_l(A)$, and so μ_l is onto.

(ii) Suppose μ_r is onto. Then $\mu_r(A) = M_r(A)$ and for each $T \in M_r(A)$, there exists some $a \in A$ such that. $T = R_a$. Hence $I = R_e$ for some $e \in A$ (since $I \in M_r(A)$) and $I(a) = R_e(a)$ for all $a \in A$. So $a = ae$ for all $a \in A$ and A has a right identity e.

Conversely, suppose that A has a right identity e. To show that μ_r is onto, let $T \in M_r(A)$. Then ,for any $a \in A$,

$$T(a) = T(ae) = aT(e) = R_{T(e)}(a);$$

that is, $T = R_{T(e)} = \mu_r(T(e)) \in \mu_r(A)$. Hence $M_r(A) \subseteq \mu_r(A)$, and so μ_r is onto. ■

As a particular case of the above, we isolate the following result for further use.

Theorem 2.2.8. *Let A be an algebra with identity e. Then*

(i) $M(A) \subseteq \{L_x : x \in A\} \cap \{R_x : x \in A\}$.

(ii) If A is also commutative, then

$$M(A) = \{L_x : x \in A\} = \{R_x : x \in A\}$$

Proof. (i) Let $T \in M(A)$. Then, for any $a \in A$,

$$\begin{aligned} T(a) &= T(ea) = (Te)a \\ &= L_{Te}(a); \end{aligned}$$

hence

$$T = L_{Te} \in \{L_x : x \in A\}.$$

Again, for any $a \in A$,

$$\begin{aligned} T(a) &= T(ae) = a(Te) \\ &= R_{Te}(a); \end{aligned}$$

hence

$$T = R_{Te} \in \{R_x : x \in A\}.$$

(ii) Suppose A is also commutative. Since A contains an identity, A is faithful and so

$$\{L_x : x \in A\} \subseteq M(A),\ \{R_x : x \in A\} \subseteq M(A).$$

Hence, by part (i),

$$M(A) = \{L_x : x \in A\} \cap \{R_x : x \in A\}.$$

Since A is commutative, clearly $L_x = R_x$ for all $x \in A$. Hence

$$M(A) = \{L_x : x \in A\} = \{R_x : x \in A\}.$$

■

Note. If A is without identity ,then $M(A) \neq \{L_x : x \in A\}$ in general.

2.3. Continuous Multipliers on Banach Algebras

In this section, we present the results of Wang [26], Johnson [15, 16], and Busby [5] regarding continuous and linear multipliers (see also Larsen [19]).

Theorem 2.3.1. *Let A be a Banach algebra.*

(a) If A has a left (resp. right) bounded approximate identity, then any left (resp. right) multiplier on T on A is linear and continuous.

(b) If A is faithful, then *any multiplier T* on *A is linear and continuous*

Proof. (a) Let T be a left multiplier on A. By the factorization theorem (see Section 1.4), considering A as a left A-module over itself, given any sequence $\{x_n\} \subseteq A$ with $x_n \longrightarrow 0$, there exist $y \in A$ and $\{z_n\} \subseteq A$ with $z_n \longrightarrow 0$ such that $x_n = yz_n$. To show that T is linear, let $x_1, x_2 \in A$ and $\alpha, \beta \in \mathbb{K}$. Taking $\{x_n\} = \{x_1, x_2, 0, 0, ...\}$, there exist $y, z_1, z_2 \in A$ such that $x_1 = yz_1, x_2 = yz_2$. So

$$\begin{aligned} T(\alpha x_1 + \beta x_2) &= T(y(\alpha z_1 + \beta z_2)) = T(y)(\alpha z_1 + \beta z_2) \\ &= \alpha T(yz_1) + \beta T(yz_2) = \alpha T(x_1) + \beta(x_2); \end{aligned}$$

hence T is linear. To show that T is continuous, it suffices to show that if $\{x_n\} \subseteq A$ with $x_n \longrightarrow 0$, then $T(x_n) \longrightarrow 0$. Using again the factorization theorem, we can write $x_n = yz_n$, where $x \in A$ and $\{z_n\} \subseteq A$ with $z_n \longrightarrow 0$. Then

$$\|T(x_n)\| = \|T(yz_n)\| = \|T(y)z_n\| \leq \|T(y)\|.\|z_n\| \longrightarrow \|T(y)\|.0 = 0.$$

Thus T is continuous.

In the case of T a right multiplier, we can apply the right hand version of the factorization theorem and obtain easily the linearity and continuity of T.

(b) Let T be a multiplier on A. We first show that T is linear. Let $x, y \in$ A and $\alpha, \beta \in \mathbb{K}$. Then, for any $z \in A$,

$$\begin{aligned} z[T(\alpha x + \beta y)] &= T(z)[\alpha x + \beta y] \\ &= \alpha T(z)x + \beta T(z)y \\ &= \alpha z.T(x) + \beta z.Ty \\ &= z[\alpha T(x) + \beta Ty] \end{aligned}$$

Since A is faithful,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

and so T is linear. To show that T is continuous, we apply the closed graph theorem. Let $\{x_n\} \subseteq A$ with $x_n \to x \in A$ and $Tx_n \to y \in A$. Then, for any $z \in A$,

$$\begin{aligned} \|z(Tx - y)\| &\leq \|zTx - zTx_n\| + \|zTx_n - zy\| \\ &= \|(Tz)x - (Tz)x_n\| + \|z(Tx_n - y\| \\ &\leq \|Tz\|\,\|x - x_n\| + \|z\|\,\|Tx_n - y\| \to 0. \end{aligned}$$

Hence $z(Tx - y) = 0$. Since A is faithful, $Tx = y$. Therefore, by the closed graph theorem (see Section 1.2), T is continuous. ■

Notation. If A is a Banach or topological algebra, then we shall always assume that all multipliers in $M_l(A)$, $M_r(A)$ and $M(A)$ are linear and continuous.

Theorem 2.3.2. *Let A be a Banach algebra.Then $M_l(A)$ and $M_r(A)$ are closed subalgebras of $(BL(A), u)$ (with u the topology induced by the operator norm $\|.\|$) which contain the identity I. In particular, $(M_l(A), u)$ and $(M_r(A), u)$ are Banach algebras with identity I.*

Proof. We have already shown that $M_l(A)$ and $M_r(A)$ are algebras with identity I. To show that $M_l(A)$ is closed in $(BL(A), u.)$, let $T \in BL(A)$ with $T \in \overline{M_l(A)}$. Then there exists a sequence $\{T_n\} \subseteq M_l(A)$ such that $T_n \xrightarrow{u} T$. Then, for any $x, y \in A$,

$$T(xy) = \lim_{n\to\infty} T_n(xy) = \lim_{n\to\infty} T_n(x)y = T(x)y.$$

Hence $T \in M_l(A)$ and $M_l(A)$ is u-closed.

Thus $(M_l(A), u)$ is a Banach algebra with identity. Similarly, $(M_r(A), u)$ is a also Banach algebra with identity ■

Theorem 2.3.3. *Let A be a faithful Banach algebra.Then $M(A)$ is a closed commutative subalgebra of $(BL(A), u)$ which contains the identity. In particular, $M(A)$ is a commutative Banach algebra with identity I.*

Proof. We have already shown that $M(A)$ is a commutative algebra with identity I. To show that $M(A)$ is closed in $(BL(A), u.)$, let $T \in BL(A)$ with $T \in \overline{M(A)}^u$. Then there exists a sequence $\{T_n\} \subseteq M(A)$ such that $T_n \xrightarrow{u} T$. Then, for any $x, y \in A$,

$$\begin{aligned}
\|x(Ty) - (Tx)y\| &\leq \|x(Ty) - x(T_ny)\| + \|x(T_ny) - (Tx)y\| \\
&= \|x(T - T_n)(y)\| + \|T_n(x)y - (Tx)y\| \\
&= \|x(T - T_n)(y)\| + \|(T_n - T)(x)y\| \\
&\leq \|x\| \, \|T - T_n\| \, \|y\| + \|T_n - T\| \, \|x\| \, \|y\| \to 0.
\end{aligned}$$

And so $x(Ty) = (Tx)y$. Alternatively,

$$x(Ty) = x(\lim_{n\to\infty} T_ny) = \lim_{n\to\infty} x(T_ny) = \lim_{n\to\infty} T_n(x)y = (Tx)y.$$

Hence $T \in M(A)$ and $M(A)$ is u-closed.

Thus $M(A)$ is a closed commutative subalgebra of $BL(A)$ with identity. ■

Definition 2.3.4. Let A be a faithful normed algebra. The *strong operator topology* or the *strict topology* s on $M(A)$ is defined as the locally convex topology generated by a family $\{p_x : x \in A\}$ of seminorms on $M(A)$, where

$$p_x(T) = \|Tx\|, \quad T \in M(A).$$

Note that, if $\{T\alpha\} \subseteq M(A)$, then $T\alpha \xrightarrow{s} T \Leftrightarrow$ for each $x \in A, T\alpha x \to Tx$ in A (i.e $\|T\alpha x - Tx\| \to 0$). Clearly, $s \leq u$. Similarly, the s-topology can also be defined on $M_l(A)$ and $M_r(A)$.

Theorem 2.3.5. *Let A be a Banach algebra.*

(a) *If A is faithful, then $(M(A), s)$ is complete.*

(b) *$(M_l(A), s)$ and $(M_r(A), s)$ are complete.*

Proof. (a) Let $\{T\alpha : \alpha \in I\}$ be an s-Cauchy net in $M(A)$. Then , for any $x \in A$,

$$\|T\alpha x - T\beta x\| = \|(T\alpha - T\beta)(x)\| \leq \|T\alpha - T\beta\| \, \|x\|,$$

and so $\{T\alpha x : \alpha \in I\}$ is a Cauchy net in A. Since A is complete , for each $x \in A, Tx = \lim_\alpha T\alpha x \in A$. Clearly, $T : A \to A$ is well-defined. Now, for any $x \in A$,

$$p_x(T\alpha - T) = \|(T\alpha - T)(x)\| = \|T\alpha(x) - T(x)\| \to 0.$$

Hence $T\alpha \xrightarrow{s} T$. Next $T \in M(A)$, as follows. For any $x, y \in A$,

$$\begin{aligned}
\|x(Ty) - (Tx)y\| &\leq \|x(Ty) - x(T\alpha y)\| + \|x(T\alpha y) - (Tx)y\| \\
&= \|x(Ty - T\alpha y)\| + \|(T\alpha x)y - (Tx)y\| \\
&= \|x(Ty - T\alpha y)\| + \|(T\alpha x - Tx)y\| \\
&\leq \|x\| \, \|Ty - T\alpha y\| + \|T\alpha x - Tx\| \, \|y\| \to 0,
\end{aligned}$$

and so $x(Ty) = (Tx)y$. [Alternatively,

$$\begin{aligned}
x(Ty) &= x(\lim_\alpha T\alpha y) = \lim_\alpha x(T\alpha y) \\
&= \lim_\alpha (T\alpha x)y = (Tx)y.]
\end{aligned}$$

Therefore $T \in M(A)$.Thus $T\alpha \xrightarrow{s} T \in M(A)$, and so $(M(A), s)$ is complete.

(b) This proof is similar to that of part (a). ■

Theorem 2.3.6. *Let A be a faithful commutative Banach algebra, and let $\mu : A \to M(A)$ be the mapping defined by*

$$\mu(x) = L_x = R_x, \quad x \in A.$$

Then :

(i) μ is well-defined.

(ii) $\mu(A) = \{L_x : x \in A\} = L_A$ is an ideal in $M(A)$.

(iii) μ is a linear isomorphism of A onto $\mu(A)$.

(iv) $\|\mu(x)\| \leq \|x\|$ for all $x \in A$; *hence μ is continuous on A.*

(v) If A possesses a minimal approximate identity, then μ is an isometry, i.e. $\|\mu(x)\| = \|x\|$ for all *$x \in A$. In this case, $\mu(A)$ is a closed ideal in $M(A)$*

Proof. (i) Since A is commutative, $L_x, R_x \in M(A)$ and further $L_x = R_x$. Hence μ is well-defined.

(ii) To show that $\mu(A) = L_A$ is an ideal in $M(A)$, let $L_x \in L_A$ and $T \in M(A)$. Then ,for any $c \in A$,

$$(L_xT)(c) = L_x(Tc) = x(Tc) = T(x)c = L_{Tx}(c)$$

and

$$(TL_x)(c) = T(L_x(c)) = T(xc) = T(x)c = L_{Tx}(c).$$

Hence

$$L_xT = L_{Tx} \in L_A, \quad TL_x = L_{Tx} \in L_A.$$

Thus $\mu(A) = L_A$ is a two-sided ideal in $M(A)$.

(iii) To show that μ is linear, let $a, b \in A$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\begin{aligned} \mu(\alpha a + \beta b) &= L_{\alpha a + \beta b} = L_{\alpha a} + L_{\beta b} \\ &= \alpha L_a + \beta L_b = \alpha\mu(a) + \beta\mu(b) \end{aligned}$$

To show that μ is homomorphism, let $x, y \in A$. Then $\mu(xy) = L_{xy} = L_xL_y = \mu(x)\mu(y)$.

To show that μ is one-one, let $x_1, x_2 \in A$ with $\mu(x_1) = \mu(x_2)$. Then, for any $c \in A$,

$$\mu(x_1)(c) = \mu(x_2)(c), L_{x_1}(c) = L_{x_2}(c), x_1c = x_2c.$$

Since A is faithful , $x_1 = x_2$. Hence μ is one-one. To show that $\mu : A \to \mu(A)$ is onto, let $L_x \in \mu(A)$, where $x \in A$. Then $\mu(x) = L_x$. Hence μ is onto .

Thus μ is a linear isomorphism of A onto $\mu(A)$.

(iv) $\|\mu(x)\| \leq \|x\|$, as follows. For any $x \in A$,

$$\begin{aligned} \|\mu(x)\| &= \|L_x\| = \sup_{\substack{a\in A \\ \|a\|\leq 1}} \|L_x(a)\| = \sup_{\substack{a\in A \\ \|a\|\leq 1}} \|xa\| \\ &\leq \sup_{\substack{a\in A \\ \|a\|\leq 1}} \|x\| \, \|a\| = \|x\| \, . \end{aligned}$$

Hence μ is bounded on A or , equivalently ,continuous on A.

(v) We now show that μ is an isometry i.e.$\|\mu(x)\| = \|x\|$. Suppose A has a "minimal " approximate identity $\{e\alpha\}$. As shown in (iv),$\|\mu(x)\| \leq \|x\|$. To show that $\|\mu(x)\| \geq \|x\|$, let $\varepsilon > 0$ be given. Since $\|e\alpha\| \to 1, \exists\, \alpha_0 \in I$ such that $|\|e\alpha\| - 1| < \varepsilon$ for all $\alpha \geq \alpha_0$. In particular, for all $\alpha \geq \alpha_0$,

$$\|e\alpha\| < 1 + \varepsilon, \left\| \frac{e\alpha}{1+\varepsilon} \right\| < 1; \qquad 0.2$$

hence

$$\|\mu(x)\| = \sup_{\substack{a\in A \\ \|a\|\leq 1}} \|xa\| \geq \left\| x \frac{e\alpha}{1+\varepsilon} \right\| \qquad 0.3$$

$$\Rightarrow \|\mu(x)\| \geq \lim_{\alpha} \left\| \frac{xe\alpha}{1+\varepsilon} \right\| = \frac{\|\lim xe\alpha\|}{1+\varepsilon} = \frac{x}{1+\varepsilon}.$$

Since ε is arbitrary, $\|\mu(x)\| \geq \|x\|$. Thus

$$\|\mu(x)\| = \|x\| \, .$$

Note. In (v), the proof of the part $\|\mu(x)\| \geq \|x\|$ is more simple if A has an approximate identity $\{e\alpha\}$ with $\|e\alpha\| \leq 1$ for all $\alpha \in I$. In this case, for any $\alpha \in I$,

$$\|\mu(x)\| = \sup_{\substack{a\in A \\ \|a\|\leq 1}} \|xa\| \geq \|xe\alpha\|$$

$$\Rightarrow \|\mu(x)\| \geq \lim_{\alpha} \|xe\alpha\| = \left\| \lim_{\alpha} xe\alpha \right\| = \|x\| \, .$$

■

The following example, due to Wang [26, p. 1138], shows that $\mu(A)$ need not be u-closed in $M(A)$, in general.

Example. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with $\varphi(x) \geq 1$ ($x \in \mathbb{R}$) and $\lim_{x \longrightarrow +\infty} \varphi(x) = \lim_{x \longrightarrow -\infty} \varphi(x) = +\infty$. Define $A = \{f \in C(\mathbb{R},\mathbb{C}) : \| f \| = \| f \|_{\varphi} = \sup_{x \in \mathbb{R}} | \varphi(x) f(x) | < \infty\}$. Then A is a faithful Banach algebra and with the operator norm dense in $C_0(\mathbb{R}, \mathbb{C})$, and so A is not complete in the operator norm. (For detail, see Larsen [19].)

Theorem 2.3.7. *Let A be a faithful Banach algebra. Then the following are equivalent*

(i) A is commutative.

(ii) $\{L_x : x \in A\} \subseteq M(A)$.

(iii) $\{R_x : x \in A\} \subseteq M(A)$.

(iv) $M(A)$ is a maximal commutative subalgebra of $BL(A)$.

Proof. $(i) \Rightarrow (ii)$ Suppose A is commutative and let $x \in A$. Then for any $a, b \in A$,

$$aL_x(b) = a(x.b) = (ax)b = (xa)b = L_x(a)b,$$

and so $L_x \in M(A)$. Hence $\{L_x : x \in A\} \subseteq M(A)$.

$(i) \Rightarrow (iii)$, Suppose A is commutative , and let $x \in A$. Then for any $a, b \in A$,

$$\begin{aligned} aR_x(b) &= a(bx) = a(xb) \\ &= (ax)\, b = R_x\,(a)\, b. \end{aligned}$$

and so $R_x \in M(A)$. Hence $\{R_x : x \in A\} \subseteq M(A)$.

$(ii) \Rightarrow (i)$ Suppose $\{L_x : x \in A\} \subseteq M(A)$. To show A is commutative , let $a, b \in A$. Then, for any $c \in A$,

$$(ab)\, c = a\,(bc) = aL_b\,(c) = L_b\,(a)\, c = (ba)\, c.$$

Since A is faithful , $ab = ba$. Hence A is commutative.

$(iii) \Rightarrow (i)$ Suppose $\{R_x : x \in A\} \subseteq M(A)$. To show A is commutative, let $a, b \in A$. Then for any $c \in A$,

$$c\,(ab) = (ca)\,b = R_a\,(c)\,b = cR_a\,(b) = c\,(ba)\,.$$

Since A is faithful, $ab = ba$. Hence A is commutative .

$(iv) \Rightarrow (i)$ Suppose $M(A)$ is a maximal commutative subalgrbra of $BL(A)$. To show that A is commutative , we show (in view of $(ii) \Rightarrow (i)$) that $\{L_x : x \in A\} \subseteq M(A)$. Let $x \in A$. Take any $T \in M(A)$. Then, for any $c \in A$,

$$(L_xT)\,(c) = L_x\,(Tc) = x\,(Tc) = T\,(xc) = T\,(L_x\,(c)) = (TL_x)(c).$$

Consequently, $L_xT = TL_x$ and so each L_x commutes with every $T \in M(A)$. Since $M(A)$ is a maximal commutative subalgebra of $BL(A)$, we have $\{L_x : x \in A\} \subseteq M(A)$.Hence ,by $(ii) \Rightarrow (i)$, A is commutative.

$(i) \Rightarrow (iv)$ Suppose A is commutative. Then,by $(i) \Rightarrow (ii)$, $\{L_x : x \in A\} \subseteq M(A)$. Suppose $M(A)$ is not a maximal commutative subalgebra of $BL(A)$. Since $BL(A)$ contains an identity I ,by Zorn's Lemma, there exists a maximal commutative subalgebra $MC(A)$ of $BL(A)$ which properly contains $M(A)$ (i.e.$M(A) \subsetneq MC(A)$).We now show that $MC(A) \subseteq M(A)$. Let $T \in MC(A)$.Then, for any $a, b \in A$,

$$\begin{aligned} a(Tb) &= L_a(Tb) = (L_aT)\,(b) = (TL_a)\,(b) \\ &= T(L_a(b)) = T(ab); \end{aligned}$$

also

$$\begin{aligned} (Ta)b &= L_b(Ta) = (L_bT)\,(a) = (TL_b)\,(a) \\ &= T(L_b(a)) = T(ba) = T(ab). \end{aligned}$$

Hence $a(Tb) = (Ta)b$, and so $T \in M(A)$. Therefore, $MC(A) \subseteq M(A)$. This contradicts the maximality of $MC(A)$. Thus $M(A)$ is a maximal commutative subalgebra of $BL(A)$. ■

Theorem 2.3.8. *Let A be a faithful commutative Banach algebra.Then the following are equivalent:*

(i) $\{L_x : x \in A\}$ *is s-dense in* $M(A)$.

(ii) A *has an approximate identity.*

Proof. $(i) \Rightarrow (ii)$ Suppose $\{L_x : x \in A\}$ is s-dense in $M(A)$. Since $I \in M(A)$, there exists a net $\{L_{e\alpha}\} \subseteq \{L_x : x \in A\}$ such that $L_{e\alpha} \xrightarrow{s} I$. We now see that $\{e\alpha : \alpha \in I\}$ is an approximate identity for A. For any $a \in A$,

$$\lim_{\alpha} e_{\alpha}a = \lim_{\alpha} L_{e_{\alpha}}(a) = I(a) = a$$

Since A is commutative, $\{e_{\alpha}\}$ is a two-sided approximate identity for A. (Note that here we do not use the faithful property. Also, if A is not necessarily commutative, then (i) implies that A has a left approximate identity.)

$(ii) \Rightarrow (i)$ Suppose A has an approximate identity $\{e_{\alpha} : \alpha \in I\}$. To show (i), let $T \in M(A)$. We now show that $L_{T_{(e\alpha)}} \xrightarrow{s} T$. For any $a \in A$,

$$\lim_{\alpha} L_{T_{(e\alpha)}}(a) = \lim_{\alpha} T(e\alpha)a = \lim_{\alpha} e\alpha.T(a) = T(a).$$

Hence $L_{T(e\alpha)} \xrightarrow{s} T$. Since $\{L_{T_{(e\alpha)}} : \alpha \in I\} \subseteq \{L_x : x \in A\}$, we have that $\{L_x : x \in A\}$ is s-dense in $M(A)$. ■

Note. In the above theorem, the commutativity of A is assumed to ensure that $\{L_x : x \in A\} \subseteq M(A)$.

Theorem 2.3.9. *Let* A *be a faithful commutative Banach algebra,* $L_A = \{L_x : x \in A\}$ *and* $B = \overline{L_A}^{s}$, *the strong operator closure of* L_A *in* $M(A)$. *Then*

(i) B *is a faithful commutative Banach algebra.*

(ii) $M(A) \subseteq M(B)$.

(iii) If $A = \overline{A^2}^{u.}$, *the norm closure of* A^2, *then* $M(A) = M(B)$.

Proof. (i) To show that B is an algebra, we first note that L_A is an algebra. [Let $L_x, L_y \in L_A$ and $p, q \in \mathbb{K}$. Then

$$\begin{aligned} pL_x + qL_y &= L_{px} + L_{qy} = L_{px+qy} \in L_A \\ L_xL_y &= L_{xy} \in L_A.] \end{aligned}$$

Since the closure of an algebra is an algebra, $B = \overline{L_A}^s$ is also an algebra. Since $B \subseteq M(A)$ and $M(A)$ is commutative, so B is also commutative.

Next, B is a Banach algebra i.e. B is norm closed in $(M(A), u)$. Since $s \leq u$. on $M(B)$, $\overline{B}^u \subseteq \overline{B}^s = B$. But $B \subseteq \overline{B}^u$. Hence $\overline{B}^u = B$. So B, being a norm -closed subset of the Banach algebra $(M(A), u)$, is a Banach algebra.

To show that B is faithful, let $T \in B$ such that $TS = 0$ for all $S \in B$. In particular, $TL_x = 0$ for all $x \in A$. Fix any $x \in A$. Then, for any $c \in A$,

$$TL_x(c) = 0, T(L_x(c)) = 0, T(xc) = 0, T(x)c = 0$$

Since A is faithful, $Tx = 0$. Since $x \in A$ is arbitrary, $Tx = 0$ for all $x \in A$; that is, $T = 0$. Thus B is faithful.

(ii) To show that $M(A) \subseteq M(B)$, let $T \in M(A)$. We first show that T maps B into $\dot{B}$; that is, for any $S \in B$, $TS \in B$. Let $S \in B = \overline{L_A}^s$. Then $\exists$ a net $\{L_{x\alpha} : \alpha \in I\}$ such that $L_{x\alpha} \xrightarrow{s} S$. Now, for any $a \in A$,

$$\begin{aligned} \lim_\alpha L_{Tx\alpha}(a) &= \lim_\alpha T(x\alpha)a = \lim_\alpha x\alpha(Ta) \\ &= \lim_\alpha L_{x\alpha}(Ta) = S(Ta) = (ST)(a). \end{aligned}$$

Hence $L_{Tx\alpha} \xrightarrow{s} ST$, and so $ST \in \overline{L_A}^s = B$. Next , for any $S_1, S_2 \in B$,

$$\begin{aligned} S_1(TS_2) &= (S_1T)S_2 \\ &= (TS_1)S_2. \end{aligned}$$

Hence $T \in M(B)$. Thus $M(A) \subseteq M(B)$.

(iii) Suppose $\overline{A^2}^u = A$ i.e., $\overline{A.A}^u = A$.(Note that $A.A \subseteq A$ with $A.A = A$ if A has an identity e).Then $\{L_{xy} : x, y \in A\}$ is u-dense in $L_A = \{L_z : z \in A\}$ as follows. Let $L_z \in L_A$ and $\varepsilon < 0$. Since $A \subseteq \overline{A^2}^u$, $\exists\, x, y \in A$ such that .

$$\|xy - z\| < \varepsilon.$$

Then

$$\|L_{xy} - L_z\| = \|L_{xy-z}\| \leq \|xy - z\| < \varepsilon.$$

Hence $L_{xy} \in \overline{L_A}^u$. So $\{L_{xy} : x, y \in A\}$ is u.-dense in L_A.

By part (ii), $M(A) \subseteq M(B)$. To show that $M(B) \subseteq M(A)$, let $T \in M(B)$. We first show that $TL_{xy} \in L_A$ for all $x, y \in A$. Since $T : B \to B, TL_x \in B \subseteq M(A)$; since L_A is an ideal in $M(A)$, $(TL_x)L_y \in L_A$, or $TL_{xy} \in L_A$.

We next show that $TL_z \in L_A$ for each $z \in A$. Let $z \in A$, $L_A = \overline{\{L_{xy} : x, y \in A\}}^u$,so $\exists$ a net $\{L_{x\alpha y\alpha} : \alpha \in I\}$ in $\{L_{xy} : x, y \in A\}$ such that $L_{x\alpha y\alpha} \overset{\|\cdot\|}{\to} L_z = TL_xL_y = TL_{xy} \in L_A$

Now, for any $z \in A$, define Tz by

$$(Tz)\,x = (TL_z)\,(x)\,, \qquad x \in A.$$

Then $T : A \to A$ is well-defined since $TL_z \in L_A$ for all $z \in A$. To show that $T \in M(A)$, let $z_1, z_2, \in A$. then, for any $c \in A$,

$$\begin{aligned} z_1(Tz_2)c &= z_1(TL_{z_2})(c) = L_{z_1}(TL_{z_2}(c)) = [L_{z_1}(TL_{z_2})]\,(c) \\ &= [(TL_{z_1})L_{z_2}]\,(c) = (TL_{z_1})(L_{z_2}(c)) = (TL_{z_1})(z_2c) \\ &= (Tz_1)(z_2c) = (Tz_1)z_2.c \end{aligned}$$

Since A is faithful, $z_1(Tz_2) = (Tz_1)z_2$; hence $T \in M(A)$. Thus $M(B) \subseteq M(A)$. ■

CHAPTER 3

COMMUTATIVE THEORY OF MULTIPLIERS

This chapter deals with the commutative theory of multiplier algebras. In section 3.1, it is shown that if A is a commutative semisimple Banach algebra, then each $T \in M(A)$ is determined by a continuous bounded complex-valued function on the maximal ideal space $\Delta(A)$ of A. In particular, if $A = C_0(X)$ with X a locally compact Hausdorff space, then $M(A) = C_b(X)$. Further, a linear mapping $T : A \to A$ is a multiplier $\Leftrightarrow$ it is a "Jordan multiplier" (Theorem 3.1.6). In section 3.2, we study various properties of the maximal ideal space $\Delta(M(A))$ with respect to both the Gelfand topology w^* and hull-kernel topology τ_{hk}. We consider isometric multipliers in section 3.3. The above results are mainly due to Wang [26], Birtel [1, 2], Máté [21], and Larsen [19].

3.1. Characterization of Multipliers on Commutative Banach Algebras

Theorem 3.1.1. (Wang) *Let A be a faithful commutative Banach algebra and $T \in M(A)$. Then $\exists$ a unique continuous function $\phi = \phi^T : \Delta(A) \to \mathbb{C}$ such that*

(i) $(\hat{Tx}) = \phi. \hat{x}$ $(i.e. (\hat{Tx})(f) = \phi(f) \hat{x}(f), f \in \Delta(A))(x \in A)$

(ii) $\|\phi\|_\infty \leq \|T\|$.

Proof. (i) Existence of ϕ : For each $f \in \Delta(A)$, choose $x \in A$ such that $f(x) \neq 0$ (since $f \neq 0$) and define

$$\phi(f) = \frac{(\hat{T} x)(f)}{\hat{x}\,(f)} = \frac{f(Tx)}{f(x)}.$$

Then ϕ is well-defined (i.e.ϕ is independent of x), as follows. Let $f \in \Delta(A)$ and $x, y \in A$ such that $f(x) \neq 0$, $f(y) \neq 0$.

$$(\hat{Tx})(f). \hat{y}\,(f) = f(Tx)f(y) = f[(Tx)y]$$

and

$$\hat{x}\,(f).((\hat{Ty}))(f) = f(x)f(Ty) = f(x(Ty))$$

Since

$$(Tx)y = x(Ty),$$
$$(\hat{Tx})(f).\,\hat{y}(f) = \hat{x}(f)(\hat{Ty})(f)$$

or

$$\frac{(\hat{Tx})(f)}{\hat{x}(f)} = \frac{(\hat{Ty})(f)}{\hat{y}(f)} = \phi(f).$$

Hence ϕ is independent of x.

Next, ϕ is continuous. [Recall that, for any $x \in A$, $\hat{x} : \triangle(A) \to \mathbb{C}$ satisfies $\left\|\hat{x}\right\|_\infty \leq \|x\|$ and hence $\hat{x}$ is continuous. To show ϕ is continuous, let $\{f_\alpha\} \subseteq \triangle(A)$ with $f_\alpha \to f \in \triangle(A)$. Then $\phi(f_\alpha) = \frac{(\hat{Tx})(f_\alpha)}{\hat{x}(f_\alpha)} \to \frac{(\hat{Tx})(f)}{\hat{x}(f)} = \phi(f)$, and so ϕ is continuous on $\triangle(A)$].

$\hat{Tx} = \phi\,\hat{x}$ holds.[We verify that $\left(\hat{Tx}\right)(f) = \phi(f)\,\hat{x}(f)$ is valid for all $f \in \triangle(A)$ and $x \in A$.We show that if $\hat{x}(f) = 0$, then $\left(\hat{Tx}\right)(f) = 0$. Choose $y \in A$ with $\hat{y}(f) \neq 0$. Then

$$\left(\hat{Tx}\right)(f).\,\hat{y}(f) = \hat{x}(f)\left(\hat{Ty}\right)(f) = 0.\ \left(\hat{Ty}\right)(f) = 0 \Rightarrow \left(\hat{Tx}\right)(f) = 0.$$

Hence $\left(\hat{Tx}\right)(f) = \phi(f)\,\hat{x}(f)$ is valid for all $f \in \triangle(A)$ and $x \in A$.That is , $\left(\hat{Tx}\right) = \phi.\,\hat{x}\ (x \in A)$.]

Also, ϕ is unique. [Suppose $\psi : \triangle(A) \to \mathbb{C}$ is another continuous function such that

$$\left(\hat{Tx}\right) = \psi.\,\hat{x}\ (x \in A).$$

Since also $\hat{Tx} = \phi\,\hat{x}\ (x \in A)$,for any $f \in \triangle(A)$

$$\begin{aligned}[\phi(f) - \psi(f)]\,\hat{x}(f) &= \phi(f)\,\hat{x}(f) - \psi(f)\,\hat{x}(f)\\ &= (\hat{Tx})(f) - (\hat{Tx})(f) = 0,\end{aligned}$$

or, since $\hat{x}(f) \neq 0$,

$$\phi(f) = \psi(f).$$

Hence ϕ is unique.]

(ii) ϕ is bounded (i.e.$\phi \in C_b(\,\triangle(A))$& $\|\phi\|_\infty \leq \|T\|$). [For each $m \in \triangle(A)$, put $K_f = \sup_{\substack{x\in A\\ \|x\|=1}} |f(x)|$. Then clearly $|f(x)| \leq K_f\,\|x\|$ for all $f \in \triangle(A)$. Further $0 < K_f \leq 1$, since

$$K_f = \sup_{\substack{x\in A\\ \|x\|=1}} |f(x)| \leq \sup_{\substack{x\in A\\ \|x\|=1}} \|f\|\,\|x\| = \|f\| \leq 1.$$

Now, for each $x \in A$,

$$\left|\phi(f)\stackrel{\wedge}{x}(f)\right| = \left\|\stackrel{\wedge}{(Tx)}(f)\right\| \leq K_f \|Tx\| \leq K_f \|T\| \|x\| \ or \ |\phi(f)| \leq \frac{K_f \|T\| \|x\|}{\left|\stackrel{\wedge}{x}(f)\right|}$$

$$\Rightarrow \quad |\phi(f)| \leq \inf_{\substack{x \in A \\ \|x\|=1}} \frac{K_f \|T\| \|x\|}{\left|\stackrel{\wedge}{x}(f)\right|} = \inf_{\substack{x \in A \\ \|x\| \leq 1}} \frac{K_f \|T\|}{\left|\stackrel{\wedge}{x}(f)\right|}$$

$$= \frac{K_f \|T\|}{\sup\limits_{\substack{x \in A \\ \|x\|=1}} \left|\stackrel{\wedge}{x}(f)\right|} = \frac{K_f \|T\|}{K_f} = \|T\|,$$

Hence ϕ is bounded and $\|\phi\|_\infty = \sup\limits_{f \in \triangle(A)} |\phi(f)| \leq \|T\|$.] ■

Notation 3.1.2 For any commutative Banach algebra A, let $C_b(\triangle(A))$ be the the normed algebra of all bounded continuous functions $\phi : \triangle(A) \to \mathbb{C}$ with norm

$$\|\phi\| = \sup_{\substack{\mu \in \triangle(A) \\ \|\mu\| \leq 1}} |\phi(\mu)|, \phi \in C_b(\triangle(A))$$

and

$$m(A) = \left\{\phi \in C_b(\triangle(A)) : \phi \stackrel{\wedge}{A} \subseteq \stackrel{\wedge}{A}\right\}.$$

Corollary 3.1.3. *Let A be a faithful commutative Banach algebra. Then $\exists$ a continuous linear mapping $h : M(A) \to m(A)$ such that*

(i) $h(T) = \phi = \phi^T$, where $\stackrel{\wedge}{(Tx)} = \phi. \stackrel{\wedge}{x}$ $(x \in A)$ and $\phi \stackrel{\wedge}{A} \subseteq \stackrel{\wedge}{A}$.

(ii) h is onto if A is semi-simple.

(iii) h is one-one if A is semi-simple.

Proof. By the above theorem, for each $T \in M(A)$, $\exists$ a function $\phi = \phi^T \in C_b(\triangle(A)$ such that

$$\stackrel{\wedge}{(Tx)} = \phi. \stackrel{\wedge}{x}, \qquad (x \in A).$$

(i) Define $h : M(A) \to m(A)$ by

$$h(T) = \phi = \phi^T, \qquad T \in M(A).$$

where ϕ is given.Then ,for any $x \in X$,

$$\phi. \stackrel{\wedge}{x} = \stackrel{\wedge}{Tx} \subseteq \stackrel{\wedge}{A};$$

(since $Tx \in A$); hence $\phi. \stackrel{\wedge}{A} \subseteq \stackrel{\wedge}{A}$.

(ii) h is onto. [Let $\phi \in m(A)$. Then ,by defintion of $m(A)$, $\phi.\ \hat{A} \subseteq \hat{A}$. Hence ,for each $x \in A$, $\phi.\ \hat{x} \in \hat{A}$. So we have a mapping $T: A \to A$ such that $\phi\ \hat{x} = \widehat{(Tx)}, x \in A$. To show that $T \in M(A)$, let $x, y \in A$. Then for any $f \in \triangle(A)$,

$$\begin{aligned} \hat{T}\,(xy)(f) &= [\phi\ \widehat{xy}](f) = \phi(f)\ \widehat{xy}\ (f) = \phi(f)f(xy) = \phi(f)(f(x)fy) \\ &= (\phi(f).\ \hat{x}\ (f))\ \hat{y}\ (f) = (\phi\ \hat{x})(f)\ \hat{y}\ (f) = \widehat{Tx}\ (f)\ \hat{y}\ (f). \end{aligned}$$

Hence $T\widehat{(xy)} = \widehat{Tx}\ .\ \hat{y} = \widehat{(Tx.y)}$. Since A is semi-simple, the map $x \to \hat{x}$ is one-one and so $T(xy) = (Tx)y$. Since A is commutative, $(Tx)y = T(xy) = T(yx) = Ty.x = x.Ty$. Thus $T \in M(A)$ and is onto.]

(iii) Suppose A is semi-simple. To show that h is one-one ,let $T_1, T_2 \in M(A)$ with $h(T_1) = h(T_2)$. Then, since A is semi-simple, the map $x \to \hat{x}$ is 1-1 and so

$$\begin{aligned} \phi^{T_1} &= \phi^{T_2} \Rightarrow \phi^{T_1}.\ \hat{x}\ \forall x \in A \Rightarrow \widehat{T_1 x} = \widehat{T_2 x}\ \forall x \in A. \\ &\Rightarrow T_1 x = T_2 x \forall x \in A \\ &\Rightarrow T_1 = T_2. \end{aligned}$$

Thus h is one -one. ■

Note. In general, the mapping $h: M(A) \to m(A)$ is not an isometry. We next see that if A is a "supremum-norm algebra " (i.e.$\|x\| = \left\|\hat{x}\right\|_\infty$), then h is an isometry.

Theorem 3.1.4. (Wang) *Let A be a semi-simple commutative supremum-norm algebra.Then the map $h: M(A) \to m(A)$ is an isometry i.e. if $T \in M(A)$ and $\phi \in m(A)$ is such that $\widehat{Tx} = \phi.\ \hat{x}$ $(x \in A)$, then $\|h(T)\| = \|T\|$ i.e. $\|\phi\|_\infty = \|T\|$.*

Proof. We have already proved in Theorem 3.1.1 that $\|\phi\|_\infty \leq \|T\|$. We now show that $\|T\| \leq \|\phi\|_\infty$. Since A is supremum-norm algebra, $\|x\| = \left\|\hat{x}\right\|_\infty$. Then

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \|Tx\| = \sup_{\|\hat{x}\|_\infty=1} \left\|\widehat{Tx}\right\|_\infty = \sup_{\|\hat{x}\|_\infty=1} \left\|\phi.\ \hat{x}\right\|_\infty \\ &\leq \sup_{\|\hat{x}\|_\infty=1} \|\phi\|_\infty \left\|\hat{x}\right\|_\infty = \|\phi\|_\infty . \end{aligned}$$

Thus $\|T\| = \|\phi\|_\infty$. So h is an isometry. ■

Note. If A is a faithful commutative Banach algebra, then, by Theorem 3.1.1, $M(A)$ is isomorphic into $m(A)$ via the correspondence $T \to \phi^T$, where $\hat{(Tx)} = \phi^T \hat{x}$ $(x \in A)$. If , in addition, A is semi-simple, by Corollary 3.1.3, $M(A)$ is isomorphic onto $m(A)$. So,in this case, we can consider $M(A)$ as a subspace of $C_b(\triangle(A))$, the space of all bounded continuous functions defined on the reguler maximal ideal space $\triangle(A)$ of A; $i.e.$ $M(A) \equiv m(A) \subseteq C_b(\triangle(A))$.

If A is a semi-simple commutative supremum-norm algebra, then $M(A)$ is isomorphic and isometric (hence homeomorphic) to $m(A)$.

Corollary 3.1.5. *Let A be a commutative semi-simple Banach algebra with identity e. Then*

$$T \in M(A) \Leftrightarrow \phi^T \in \widehat{A}.$$

where $\widehat{(Tx)} = \phi.\widehat{x}$ $(x \in A)$. Hence $M(A) = m(A) = \widehat{A}.$, and so $M(A)$ is trivial.

Proof. Let $T \in M(A)$. Then ,for any $x \in A$, $T(x) = T(ex) = T(e)x = L_{T(e)}(x)$. Hence $T = L_{T(e)} = L_a$, where $a = Te$. By Corollary 3.1.3., $T \in M(A) \Leftrightarrow \exists\ \phi \in m(A)$ with $\widehat{Tx} = \phi.\widehat{x}$, $x \in A$. Now, for any $x \in A$,

$$\phi\widehat{x} = \widehat{Tx} = \widehat{L_a(x)} = \widehat{(ax)} = \widehat{a}\widehat{x}.$$

for some $a \in A$. Hence $\phi = \widehat{a} \in \widehat{A}$. Thus $T \in M(A) \Leftrightarrow \phi \in \widehat{A}$. Therefore, $M(A) = m(A) = \widehat{A}$. ■

Threorem 3.1.6. (Birtel) *Let A be a semi-simple commutative Banach algebra and $T : A \to A$ a linear mapping.Then the following are equivalent :*

(i) $T \in M(A)$.

(ii) $T(M_f) \subseteq M_f$ *for all* $f \in \triangle(A)$, *where* $M_f = f^{-1}(0) = \{x \in A : f(x) = 0\}$

(iii) $T(x^2) = x(Tx)$ *for all* $x \in A$ *(i.e. T is a Jordan Multiplier).*

Proof. (i)⇒(ii) Suppose $T \in M(A)$, and let $f \in \Delta(A)$. To show that $T(M_f) \subseteq M_f$, let $y \in M_f$, Then $f(y) = 0$. Now, if $x \notin M_f$, using $T \in M(A)$ as in Theorem3.1.1,

$$\begin{aligned} f(x)f(Ty) &= f(x.Ty) = f(Tx.y) \\ &= f(Tx)f(y) = 0. \end{aligned}$$

Since $f(x) \neq 0$, we have $f(Ty) = 0$. Hence $Ty \in M_f$. This proves that $T(M_f) \subseteq M_f$.

(ii)⇒(i), Suppose $T(M_f) \subseteq M_f$ for all $f \in \Delta(A)$. To show that $T \in M(A)$, let $x, y \in A$. Take any $f \in \Delta(A)$.

Case I. Suppose either $x \in M_f$ or $y \in M_f$. Then either $f(x) = 0$ or $f(y) = 0$, also, since $T(M_f) \subseteq M_f$, either $f(Tx) = 0$ or $f(Ty) = 0$. Hence

$$f\left[x\left(Ty\right) - \left(Tx\right)y\right] = f\left(x\right).f\left(Ty\right) - f\left(Tx\right).f\left(y\right) = 0 - 0 = 0.$$

Case II. Suppose $x \notin M_f$ and $y \notin M_f$. Since M_f is maximal $\exists\ z \notin M_f$ such that $A = [z] \oplus M_f$. So $\exists \alpha, \beta \in \mathbb{K}$ and $u, v \in M_f$ such that

$$x = \alpha z + u, y = \beta z + v.$$

Then $f(u) = 0$ and $f(v) = 0$; also ,since $T\left(M_f\right) \subseteq M_f,\ f\left(Tu\right) = 0$, and $f\left(Tv\right) = 0$.Hence, since T is linear,

$$\begin{aligned} f\left[x\left(Ty\right) - \left(Tx\right)y\right] &= f\left[\left(\alpha z + u\right)\left(\beta Tz + Tv\right) - \left(\alpha Tz + Tu\right)\left(\beta z + v\right)\right] \\ &= f\left[\alpha\beta zTz + \alpha zT\left(v\right) + \beta uT\left(z\right) + uT\left(v\right) - \alpha\beta T\left(z\right)z - \alpha T\left(z\right)v - \beta T\left(u\right)z - T\left(u\right).v\right] \\ &= \alpha\beta f\left(z\right)f\left(T(\left(z\right)\right) + 0 + 0 + 0 - \alpha\beta f\left(T\left(z\right)\right)f\left(z\right) - 0 - 0 - 0 = 0 \end{aligned}$$

Hence, in each case,

$$f\left[x(Ty) - \left(Tx\right)y\right] = 0,$$

for all $f \in \Delta(A)$. Therefore

$$x(Ty) - (Tx)y \in \cap\{M_f : f \in \Delta(A)\} = rad\ (A).$$

Since A is semi-simple, $rad\ (A) = \{0\}$ and so

$$x(Ty) - (Tx)y = 0, x(Ty) = (Tx)y.$$

Thus $T \in M(A)$.

(i)⇒(iii) Suppose $T \in M(A)$, and let $x \in A$. Since A is faithful, $T \in M_l(A) = M_r(A)$. Hence

$$T(x^2) = T(x.x) = x(Tx).$$

We show that $T \in M(A)$, or equivalently, $T \in M_r(A)$ (since A is faithful and commutative). For any $x, z \in A$, by hypothesis,

$$\begin{aligned} x(Tx) + 2T(xz) + z(Tz) &= T(x^2) + 2T(xz) + T(z^2) \\ &= T(x^2 + 2xz + z^2 = T((x+z)^2) \\ &= (x+z)T(x+z) = (x+z)(Tx+Tz) \end{aligned}$$

or

$$x(Tx) + 2T(xz) + z(Tz) = x(Tx) + x(Tz) + z(Tx) + z(Tz)$$

or

$$2T(xz) = x(Tz) + z(Tx)$$

or

$$2T(xz) - x(Tz) = z(Tx) = (Tx)z$$

or

$$2TL_x(z) - L_x(Tz) = L_{Tx}(z)$$

or

$$(2TL_x - L_xT)(z) = L_{Tx}(z)$$

Hence

$$2TL_x - L_xT = L_{Tx} \in M(A)$$

Since $M(A)$ is commutative, for any $z \in A$,

$$(2TL_x - L_xT)\,L_z = L_z(2TL_x - L_xT)$$

or

$$2TL_xL_z - L_xTL_z = 2L_zTL_x - L_zL_xT.$$

Using $T(z^2) = zT(z)$,

$$2TL_xL_z(z) - L_xTL_z(z) = 2L_zTL_x(z) - L_zL_xT(z)$$

or

$$2T(xz^2) - xT(z^2) = 2zT(xz) - zx(Tz)$$

or

$$2T(xz^2) - xz(Tz) = 2zT(xz) - zx(Tz)$$

or

$$2T(xz^2) = 2z(Txz)$$

or

$$T(xz^2) = zT(xz)$$

or

$$T(z^2x) = zT(zx).$$

Replacing z by $x + z$ and x by y,

$$T[(x+z)^2y] = (x+z)T[(x+z)y]$$

or

$$T[x^2y + 2xzy + z^2y) = (x+z)[T(xy) + T(zy)]$$

or

$$T(x^2y) + 2T(xzy) + T(z^2y) = xT(xy) + xT(zy) + zT(xy) + zT(zy)$$

or

$$xT(xy) + 2T(xzy) + zT(zy) = xT(xy) + xT(zy) + zT(xy) + zT(zy)$$

or

$$2TL_xL_z(y) = L_xTL_z(y) + L_zTL_x(y),\ y \in A.$$

so

$$2TL_xL_z = L_xTL_z + L_zTL_x.$$

$$\equiv 2TL_xL_z - L_xTL_z = 2L_zTL_x - L_zL_xT$$

or

$$2L_zTL_x - L_zTL_x = L_zL_xT$$

or

$$L_zTL_x = L_zL_xT$$

Hance, for any $y \in A$,

$$L_z T L_x(y) = L_z L_x T(y)$$

or

$$zT(xy) = zxT(y), z \in A$$

Since A is faithful,

$$T(xy) = x(Ty)$$

Thus $T \in M_l(A)$ or equivalently $T \in M(A)$. ■

Recall that a Banach algebra A is said to *admit factorization* (or to be *factorable*) if , for any $z \in A$, $\exists\ x, y \in A$ such that $z = xy$ (i.e.if $A = A^2$). The famous Cohen factorization theorem states that *every commutative Banach algebra with an approximate identity is factorable* (see Section 1.4).

Theorem 3.1.7. (Birtel) *Let A be a commutative semi-simple Banach algebra. Let $T \in M(A)$ be such that $\phi^T(f) \neq 0$ ($f \in \Delta(A)$). Define*

$$x \circ y = x(Ty)(x, y \in A),$$

$$\|x\|_T = \|T\| \, \|x\| \, (x \in A).$$

Then (i) (A, o) forms a commutative algebra.

(ii) $\|.\|_T$ is a norm on A equivalent to $\|.\|$ and $A_T = (A, o, \|.\|_T)$ is a Banach algebra.

(iii) $\Delta(A) = \Delta(A_T)$; hence A_T is semi-simple.

Conversely, if A forms under distinct multiplications and equivalent norms two commutative semi-simple Banach algebras A and A_1 which have the same regular maximal ideal space (i.e.$\Delta(A) = \Delta(A_1)$ and if A is factorable, then $\exists$ a $T \in M(A)$ such that $\phi^T(f) \neq 0$ ($f \in \Delta(A)$) and $A_1 = A_T$.

Proof. Let $T \in M(A)$ be such that $\phi^T(f) \neq 0\ for\ all\ f \in \Delta(A)$.

(i) We first verify that (A, o) is a commutative algebra. Let $x, y, z \in A$ and $\alpha \in \mathbb{K}$. Then

$$(x \circ y) \circ z = [x(Ty) \circ z = x(Ty)(Tz)]$$

$$x \circ (y \circ z) = x \circ [y(Tz)] = xT[y(Tz)] = x(Ty)(Tz)$$

and so

$$(x \circ y) \circ z = x \circ (y \circ z);$$

$$\begin{aligned}
x \circ (y + z) &= xT(y + z) = x(Ty + Tz) \\
&= x(Ty) + x(Tz) = x \circ y + x \circ z; \\
\alpha(x \circ y) &= \alpha[x(Ty)] = (\alpha x)(Ty) = \alpha x \circ y, \\
\alpha(x \circ y) &= \alpha[x(Ty)] = xT(\alpha y) = x \circ (\alpha y);
\end{aligned}$$

therefore,

$$\alpha(x \circ y) = (\alpha x) \circ y = x \circ (\alpha y);$$

$$x \circ y = x(Ty) = (Tx)y = y(Tx) = y \circ x.$$

Hence (A, o) is a commutative algebra.

(ii) Since $\|.\|$ is a norm on A and $\|T\| \geq 0$ is a constant, clearly $\|x\|_T = \|T\| \, \|x\|$ defines a norm $\|\cdot\|_T$ on A. Further, $\|x\|_T = \|T\| \, \|x\|$ and $\|x\| = (1/\|T\|) \, \|x\|_T$ (with $\|T\| \neq 0$) show that $\|.\|_T$ and $\|.\|$ are equivalent norms on A. Next

$$\begin{aligned}
\|x \circ y\|_T &= \|T\| \, \|x \circ y\| = \|T\| \, \|x(Ty)\| \\
&\leq \|T\| \, \|x\| \, \|Ty\| \leq (\|T\| \, \|x\|)(\|T\| \, \|y\|) = \|x\|_T \, \|y\|_T,
\end{aligned}$$

and so $\|.\|_T$ is an algebra norm.Thus $A_T = (A, o, \|.\|_T)$ is a Banach algebra.

(iii) $\Delta(A) \subseteq \Delta(A_T)$: Let $f \in \Delta(A)$. Define $f_T = \phi(f)f$. Clearly f_T is non-zero, linear and also $f_T^{-1}(0) = f^{-1}(0)$, since

$$\begin{aligned}
y &\in f_T^{-1}(0) \Leftrightarrow f_T(y) = 0 \Leftrightarrow \phi(f).f(y) = 0 \\
&\Leftrightarrow f(y) = 0 \Leftrightarrow y \in f^{-1}(0).
\end{aligned}$$

Moreover, for any $x, y \in A$,

$$\begin{aligned}
f_T(x \circ y) &= \phi(f)f(x \circ y) = \phi(f)f(x(Ty)) \\
&= \phi(f)f(x)f(Ty) = \phi(f).f(x)(Ty)\ (f)
\end{aligned}$$

$$= \phi(f)f(x)\phi(f)\hat{y}(f) = \phi(f).\, f(x).\phi(f)f(y)$$

$$= f_T(x)f_T(y),$$

and so f_T is a m.l.f. on A_T. Since $f^{-1}(0) = f_T^{-1}(0)$, we have $f \in \Delta(A_T)$. Hence $\Delta A \subseteq \Delta(A_T)$.

A_T is semi-simple, as follows::

$$\begin{aligned} rad\ (A_T) &= \cap\{g^{-1}(0) : g \in \Delta(A_T)\} \\ &\subseteq \cap\{f^{-1}(0) : f \in \Delta(A_T)\} \\ &= rad(A) = \{0\} \end{aligned}$$

Hence $rad(A_T) = \{0\}$, and so A_T is semi-simple

Next $\Delta(A_T) \subseteq \Delta(A)$, as follows. We first note that $T \in M(A_T)$. Indeed, if $x \in A_T$,

$$T(x \circ x) = T(x(Tx)) = (Tx)(Tx) = xT(Tx) = x \circ Tx,$$

and so, by Theorem 3.1.6, $T \in M(A_T)$. Next, let $\phi_T \in m(A_T)$ denote the bounded continuous function on $\Delta(A_T)$ associated with $T \in M(A_T)$ by Theorem

$$i.e.\ \hat{Tx} = \phi^T \hat{x}\ .$$

Now, to show that $\Delta(A_T) \subseteq \Delta(A)$, let $m_T \in \Delta(A_T)$. Then

$$f_T(Tx) = (\hat{Tx})(f_T) = \phi(f_T)\hat{x}(f_T) = \phi(f_T)f_T(x)\ (x \in A).$$

Thus, if $x, y \in A_T$,

$$\begin{aligned} f_T(x)f_T(y) &= f_T(x \circ y) = f_T(x(Ty)) \\ &= f_T(T(xy)) = \phi(f_T)f_T(xy). \end{aligned}$$

The above implies that $\phi(f_T) \neq 0\ (f_T \in \Delta(A_T))$.

Define $f = (f_T)/\phi(m_T)$. Clearly f is a linear functional on A. Further, for any $x, y \in A$,

$$f(xy) = (f_T(xy))/\phi(f_T) = (f_T(x)f_T(y))/(\phi(f_T)\phi(f_T)) = f(x)f(y)$$

and so f is multiplicative. Next, $f^{-1}(0) = f_T^{-1}(0)$, since

$$\begin{aligned} y &\in f^{-1}(0) \Leftrightarrow f(y) = 0 \Leftrightarrow (f_T(y))/\phi(f_T) = 0 \\ &\Leftrightarrow f_T(y) = 0 \Leftrightarrow y \in f_T^{-1}(0). \end{aligned}$$

Hence $f_T^{-1}(0) = f^{-1}(0)$ and so $f_T \in \Delta(A)$. This proves that $\Delta(A_T) \subseteq \Delta(A)$. Thus $\Delta(A_T) = \Delta(A)$.

Conversely, suppose A forms under distinct multiplications and equivalent norms two commutative semi-simlpe Banach algebras A and A_1 with $\Delta(A) = \Delta(A_1)$. Let f_1 be a m.l.f. on $A_1 = (A, 0)$. Since $\Delta(A) = \Delta(A_1)$, $f_1 \in \Delta(A)$, and so $\exists$ a m.l.f. f on A and a non-zero constant $a(f)$ such .that. $f_1 = a(f)f$. Define $\phi : \Delta(A) \to \mathbb{K}$ by

$$\phi(f) = a(f).$$

Since A is factorable, for any $z \in A, \exists\, x, y \in A$ such that $z = xy$. Then, for each m.l.f. f on A,

$$\begin{aligned}
\phi(f)\, \hat{z}\,(f) &= \phi(f)\widehat{(xy)}\,(f) = \phi(f)\, \hat{x}\,(f)\, \hat{y}\,(f) \\
&= a(f)f(x)f(y) = a(f)(f_1(x)/a(f))(f_1(y)/a(f)) \\
&= (1/a(f))(f_1(x)f_1(y)) = [1/a(f)](f_1(x \circ y)) \\
&= f(x \circ y) = \widehat{(x \circ y)}\,(f)
\end{aligned}$$

Thus, by above,

$$\phi\, \hat{z} = \phi\, \hat{x}\hat{y} = \widehat{(x \circ y)} \in \hat{A}\ (z \in A). \qquad 0.4$$

Now define a map $T : A \to A$ given by

$$\widehat{(Tz)} = \phi\, \hat{z}, z \in A$$

Then T is well-defined by $(*)$. Further, for any $z, w \in A$,

$$(T\widehat{(zw)}) = \phi\widehat{(zw)} = \phi(\hat{z}\hat{w}) = (\phi\, \hat{z})\, \hat{w} = \widehat{Tz}\hat{w}\,.$$

Therefore, $T(zw) = (Tz)w$, and so $T \in M(A)$. Next, for any $x, y \in A$,

$$\widehat{(x \circ y)} = \phi\, \widehat{(xy)} = [T\widehat{(xy)}] = [x\widehat{(Ty)}]$$

and so $x \circ y = x(Ty)$. Hence, if $\|x\|_T = \|T\|\,\|x\|$, $A_1 = (A, o, \|.\|_T)$; that is $A_1 = A_T$. ■

Corollary 3.1.8. *Let A be a commutative semi-simple Banach algebra. Then* $M(A) = M(A_T)$.

(Recall that, for any $T \in M(A)$ with $\phi(f) \neq 0$ $(f \in \Delta(A))$, $A_T = (A, 0, \|.\|_T)$, where $x \circ y = x(Ty)$ and $\|x\|_T = \|T\|\,\|x\|$ $(x \in A)$.)

Proof. To show that $M(A) \subseteq M(A_T)$, let $S \in M(A)$. Then, for each $x \in A_T$,

$$S(\, x \circ x) = S(x(Tx)) = x.T(Sx) \; (\because S \in M(A) = M_r(A))$$

So, by Theorem 3.1.7, $S \in M(A_T)$. Hence $M(A) \subseteq M(A_T)$. To show $M(A_T) \subseteq M(A)$, let $S \in M(A_T)$. By Threorem 3.1.1, $\exists$ a $\acute{\psi} \in m(A_T)$ such that

$$\hat{Sx} = \acute{\psi}\, \hat{x} \;\; (x \in A_T).$$

By Theorem 3.1.7, $\Delta(A) = \Delta(A_T)$. Hence, if $f_T \in \Delta(A_T)$, then, as in the proof of Theorem 3.1.7, $\exists$ an $f \in \Delta(A)$ and a constant $a(f)$, such that

$$f = (1/a(f)) f_T.$$

Then, for $f_T(x) \neq 0$,

$$f_T(Sx)/(f_T(x)) = [a(f)f(Sx)] \,/\, [a(f)f(x)] = f(Sx)/f(x).$$

But

$$\begin{aligned} f_T(Sx) &= (\hat{Sx})(f_T) = \acute{\psi}(f_T)\, \hat{x}\, (f_T) \\ &= \acute{\psi}(f_T) f_T(x). \end{aligned}$$

$$f(Sx) = (f_T(Sx)/f_T(x))f(x) = [(\acute{\psi}(f)f_T(x))/T(x))]f(x) = \acute{\psi}(f)f(x)$$

or

$$(\hat{Sx})(f) = \acute{\psi}(f)\, \hat{x}\, (f).$$

Hence $\hat{Sx} = \acute{\psi}\, \hat{x}$. Now, (as in Corollary 3.1.5) for any $x, y \in A$,

$$[S(\hat{xy})] = (\hat{Sx})\, \hat{y} = [(\hat{sx})y] \; .$$

Since A is semi-simple, the map $x \rightarrow \hat{x}$ is one-one and so $S(xy) = (Sx)y$. Therefore $S \in M_l(A) = M(A)$. Thus $M(A_T) \subseteq M(A)$. ■

Corollary 3.1.9. *Let A be a commutative semi-simple Banach algebra which admits factorization. Then the following are equivalent.*

(i) A_T admits factorization.

(ii) T is invertible.

Proof. (i)$\Rightarrow$ (ii). Suppose A_T admits factorization. By the above corollary.$M(A) = M(A_T)$. Since $I \in M(A)$, $\exists\, S \in (A_T)$ such that $I = S$. Then , for any $x, y \in A$,

$$I(xy) = S(x \circ y), xy = x \circ (Sy).$$

We now show that $T^{-1} = S$. Let $z \in A$. Since A and A_T are factoroable,$\exists\, x, y, u, v \in A$ such that

$$z = xy, z = u \circ v.$$

Now

$$z = xy = x \circ Sy = S(x \circ y) = S(x(Ty)) = ST(xy) = ST(z).$$

Also

$$z = u \circ v = u(Tv) = T(uv) = T(u \circ Sv) = T(S(u \circ v) = TS(z)$$

Hence, for all $z \in A$,

$$ST(z) = TS(z) = z = I(z).$$

Therefore $ST = TS = I$, and so T is invertible with $T^{-1} = S$.

(ii)$\Rightarrow$(i) Suppose T is invertible .Then T^{-1} exists and $T^{-1} \in M(A)$. To show that A_T is factorable, let $z \in A$. (We show that $z = u \circ v$ for some $u, v \in A$). Since A is factorable, $z = xy$ for some $x, y \in A$. Then, since $T^{-1} \in M(A) = M(A_T)$,

$$\begin{aligned} z &= T^{-1}(T(z)) = T^{-1}(T(xy)) = T^{-1}[xTy] \\ &= T^{-1}(x \circ y) = (T^{-1}x) \circ y = u \circ v, \end{aligned}$$

where $u = T^{-1}x$ and $v = y$. Hence A_T is factorable. ■

3.2. Maximal Ideal Spaces of Multiplier Algebras

In this section, we study various properties of the maximal ideal space $\Delta(M(A))$ of the multiplier algebras $M(A)$ and the relation between $\Delta(A)$ and $\Delta(M(A))$ with respect to both the Gelfand topology w^* and hull-kernel topology τ_{hk}

We recall that if A is a faithful commutative Banach algebra, then

(a) $M(A)$ is a commutative Banach algebra with identity I under the operator norm

$$\|T\| = \sup_{\substack{x \in A \\ \|x\| \leq 1}} \|Tx\|$$

(b) A is isomorphic to the set $\{L_x : x \in A\} \subseteq M(A)$.

Here we first study the relation between the maximal ideal spaces $\Delta(A)$ and $\Delta(M(A))$.

Theorem 3.2.1. (Máté) *Let A be a faithful commutative Banach algebra.*

(i) If f is a non-zero multiplicative linear functional on A, then $\exists$ a unique multiplicative linear functional f'on $M(A)$ such that

$$f'(L_y) = f(y) \quad (y \in A) \quad i.e. \;\; f' = f \;\; on \;\; A.$$

(ii) If g is a multiplictive linear functional on $M(A)$, then either (a) $g(L_x) = 0$ *for all* $x \in A$ *or (b) (if $g(L_y) \neq 0$ for some $y \in A$) there is a unique m.l.f. g' on $M(A)$ such that $g = g'$.*

Proof. (i) Let f be a non-zero m.l.f on A. Then $\exists\, x \in A$ such that $f(x) \neq 0$. Define $f' : M(A) \to \mathbb{K}$ by

$$f'(T) = \frac{f(Tx)}{f(x)} \;, T \in M(A) \text{___}(1)$$

This definition is independent of x since, for any $y \in A$ with $f(y) \neq 0$

$$\begin{aligned} f'(T) &= \frac{f(Tx)}{(x)} \frac{f(y)}{(y)} = \frac{f((Tx)y)}{f(x)f(y)} \\ &= \frac{f(x(Ty))}{f(x)\mu(y)} = \frac{f(x)f(Ty)}{f(x)f(y)} = \frac{f(Ty)}{f(y)}. \end{aligned}$$

f' is linear since, for any $T, S \in M(A)$ and $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} f'(\alpha T + \beta S) &= \frac{f[(\alpha T + \beta S)(x)]}{f(x)} = \frac{\alpha f(Tx)}{f(x)} + \beta \frac{f(Sx)}{f(x)} \\ &= \alpha f'(T) + \beta f'(S). \end{aligned}$$

For any $y \in A$,

$$f'(L_y) = \frac{f(L_y(x))}{f(x)} = \frac{f(yx)}{f(x)} = \frac{f(y)f(x)}{f(x)} = f(y).$$

and so $f' = f$ on A. Next, f' is multiplicative since, for any $T, S \in M(A)$, (1) gives

$$f[(Ts)(x)] = f[T(Sx)] = f'(T)f(Sx)$$

and so

$$f'(TS) = \frac{f[(TS)(x)]}{f(x)} = \frac{f'(T)f(Sx)}{f(x)} = f'(T)f'(S).$$

To show that $f'$is unique , let f^* be any other m.l.f. on $M(A)$ such that

$$f^*(L_y) = f(y),\ y \in A.$$

Choosing any $x \in A$ with $f(x) \neq 0$, we see that , for each $T \in M(A), TL_x(a) = T(xa) = T(x)a = L_{Tx}(a), a \in A$, and so

$$\begin{aligned} f^*(T)f(x) &= f^*(T)f^*(L_x) = f^*(TL_x) \\ &= f^*(L_{Tx}) = f(Tx) = f'(T)f(x). \end{aligned}$$

Hence $f^*(T) = f'(T)$ for all $T \in M(A)$ (since $f(x) \neq 0$), and so $f^* = f'$. Thus f' is unique

(ii) Let g be a m.l.f. on $M(A)$. If (a) holds, the result is proved. We now show that if (a) does not hold, then (b) holds. Suppose $g(L_y) \neq 0$ for some $y \in A$. Define $f : A \to \mathbb{K}$ by

$$f(x) = g(L_x), \quad x \in A.$$

Then f is a m.l.f.on A since, for any $x, z \in A$ and $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} f(\alpha x + \beta z) &= g(L_{\alpha x + \beta z}) = g(\alpha L_x + \beta L_z) \\ &= \alpha g(L_x) + \beta g(L_z) = \alpha f(x) + \beta f(z), \end{aligned}$$

$$f(xz) = g(L_{xz}) = g(L_x L_z) = g(L_x)g(L_z) = f(x)f(z)$$

Now define $f' : M(A) \to \mathbb{K}$ by

$$f'(T) = \frac{f(Ty)}{f(y)},\ T \in M(A).$$

Then, exactly as in part (i) , f' is a unique m.l.f. on $M(A)$. Further, taking $f^* = g$ in the proof of (i) , $f'(T) = g(T)$ for all $T \in M(A)$; hence $f' = g$. This proves(b). ■

Note. Let A be a faithful commutative Banach algebra. Consider the ideal $L_A = \{L_x : x \in A\}$ of $M(A)$. Let

$\Phi(A)$ =the set of all maximal moduler ideals of A,

$\Delta(A)$ =the set of all non-zero m.l.f.on A,

and let

$\Phi'(M(A)) = \{M' \in \Phi(M(A) : L_A \nsubseteq M'\}$

$\Delta'(M(A)) = \{f' \in \Delta(M(A)) : f' \neq 0 \text{ on } L_A\} = \{f' \in \Delta(M(A) : L_A \nsubseteq f'^{-1}(0)\}$

By the above theorem, we have:

Corollary 3.2.2. *The correspondence $f \to f'$ defines a bijective mapping of $\Delta(A)$ onto $\Delta'(M(A))$; or equivalently, the correspondence $M \to M'$ defines a bijective mapping of $\Phi(A)$ onto $\Phi'(M(A))$, where $M = f^{-1}(0)$ and $M' = f'^{-1}(0)$.*

Definition 3.2.3. Let B be a commutative Banach algebra. For any ideal I in B, we define the hull of I by

$$H(I) = \{M \in \Phi(B) : I \subseteq M\},$$

or equivalently $H_\Delta(I) = \{f \in \Delta(B) : I \subseteq f'^{-1}(0)\}$.

Theorem 3.2.4. (Wang) *Let A be a faithful commutative Banach algebra, and let*

$$H_\Delta(L_A) = \{f' \in \Delta(M(A)) : L_A \subseteq f'^{-1}(0)\}.$$

(i) $\Delta(M(A)) = H_\Delta(L_A) \cup \Delta'(M(A))$.

(ii) $(\Delta(A), w^*)$ is homeomorphic to the open set $(\Delta'(M(A)), w^*)$ under the correspondence $f \to f'$.

(iii) $(H_\Delta(L_A), w^*)$ is compact.

Proof. (i) By definitions of $\Delta'(M(A))$ and $H(L_A)$,

$$\begin{aligned} H_\Delta(L_A) \cup \Delta'(M(A)) &= \{f' \in \Delta(M(A)) : L_A \subseteq f'^{-1}(0)\} \\ \cup\{f' &\in \Delta(M(A)) : L_A \nsubseteq f'^{-1}(0)\} \\ &= \Delta(M(A)). \end{aligned}$$

(ii) By Theorem 3.2.1 the map $\theta : f \to f'$ of $\Delta(A)$ into $\Delta'(M(A))$ is a bijection (i.e. one-one and onto).

Now $\theta : (\Delta(A), w^*(A^*, A)) \to (\acute{\Delta}(M(A)), w^*(M(A)^*, M(A)))$ is continuous, as follows. Let $\{f_\alpha\} \subseteq (A)$ with $f_\alpha \xrightarrow{w^*} f \in \Delta(A)$. Then

$$f_\alpha(a) \to f(a) \text{ for all } a \in A. _____(1)$$

To show that $\theta(f_\alpha) \xrightarrow{w^*} \theta(f)$ (*i.e.* $f'_\alpha \xrightarrow{w^*} f'$) in $\Delta'(M(A))$, let $T \in M(A)$. Choose $x \in A$ with $f(x) \neq 0$. Then, by (1),

$$f'_\alpha(T) = \frac{f_\alpha(Tx)}{f_\alpha(x)} \to \frac{f(Tx)}{f(x)} = f'(T).$$

Hence $\theta(f_\alpha) \xrightarrow{w^*} \theta(f)$, and so θ is continuous.

Next, $\theta^{-1} : (\Delta'(M(A)), w^*(M(A)^*, M(A)) \to (\Delta(A), w^*(A^*, A))$ is continuous, as follows. Let $\{f'_\alpha\} \subseteq \Delta'(M(A))$ with $f'_\alpha \xrightarrow{w^*} f'$ in $\Delta'(M(A))$. Then

$$f'_\alpha(T) \to f'(T) \text{ for all } T \in M(A). ____(2)$$

To show that $\theta^{-1}(f'_\alpha) \xrightarrow{w^*} \theta^{-1}(f')$ (*i.e.* $f_\alpha \xrightarrow{w^*} f$) in $\Delta(A)$, let $x \in A$. Then, since $L_x \in L_A \subseteq M(A)$ and $g'(L_x) = g(x)$ $(g \in \Delta(A))$, by (2),

$$f_\alpha(x) = f'_\alpha(L_x) \to f'(L_x) = f(x).$$

Hence $\theta^{-1}(f'_\alpha) \xrightarrow{w^*} \theta^{-1}(f')$, and so θ^{-1} is continuous. Thus θ is a homeomorphism of $\Delta(A)$ onto $\Delta'(M(A))$.

(iii) Since $M(A)$ is a commutative Banach algebra with identity I, $\Delta(M(A))$ is w^*–compact. Now $H_\Delta(L_A)$ is w^*–closed in $\Delta(M(A))$ since

$$\begin{aligned} H_\Delta(L_A) &= \{f' \in \Delta(M(A)) : L_A \subseteq f'^{-1}(0)\} \\ &= \{f' \in \Delta(M(A)) : f'(L_A) = 0\} \\ &= \bigcap_{x \in A} \{f' \in \Delta(M(A)) : f'(L_x) = 0\}, \end{aligned}$$

which is an intersection of closed sets. Thus $H_\Delta(L_A)$ is w^*-compact. ■

Note. The above correspondence between $\Delta(A)$ and $\Delta'(M(A))$ can also be described between $\Phi(A)$ and $\Phi'(M(A))$, as follows.

For each $M' \in \Phi'(M(A)) = \{M' \in \Phi(M(A)) : L_A \not\subseteq M'\}$, we can associate $M \in \Phi(A)$ by setting

$$M = \{y \in A : L_y \in M' \cap L_A\},$$

and conversely, note that

$$\begin{aligned} M &= \{y \in A : L_y \in f'^{-1}(0) \cap L_A\} \\ &= \{y \in A : y \in f^{-1}(0) \cap A\} = f^{-1}(0). \end{aligned}$$

In this case the corresponding hull of L_A is

$$H_\Phi(L_A) = \{M' \in \Phi(M(A)) : L_A \subseteq M'\}.$$

With this terminology, the parts (i)-(iii) of Theoerem 3.3.4 may be stated as

(i) $\Phi(M(A)) = H_\Phi(L_A) \cup \Phi'(M(A))$

(ii) $\Phi(A)$ is homeomoephic to the open set $\Phi'(M(A))$ under the correspondence $M \to M'$.

(iii) $H_\Phi(L_A)$ is compact.

Corollary 3.2.5. *Let A be a faithful commutative Banach algebra. Then, for each $x \in A$, the Gelfand transform $\overset{\wedge}{L_x}$ of L_x vanishes identically on $\Delta(M(A))\backslash\Delta'(M(A)) = H_\Delta(L_A)$ (i.e. $\overset{\wedge}{L_x}$ $(H_\Delta(L_A)) = \{0\}$).*

Proof.

$$\begin{aligned} f' &\in H_\Delta(L_A) \Rightarrow L_A \subseteq f'^{-1}(0), \\ &\Rightarrow L_x \in f'^{-1}(0) \quad \text{for all } x \in A, \\ &\Rightarrow f'(L_x) = 0 \quad \text{for all } x \in A, \\ &\Rightarrow \overset{\wedge}{L_x}(f') = 0 \quad \text{for all } x \in A. \end{aligned}$$

Hence, for each $x \in A$, $\overset{\wedge}{L_x}$ $(H_\Delta(L_A)) = \{0\}$. ■

Corollary 3.2.6. *Let A be a semi-simple commutative Banach algebra. Then $M(A)$ is semi-simple.*

Proof. Since A is semi-simple,

$$rad\,(A) = \cap\{f^{-1}(0) : f \in \Delta(A)\} = \{0\}$$

Now

$$\begin{aligned}
rad\,(M(A)) &= \cap\{M_{f'} : f' \in \Delta(M(A))\} \\
&\subseteq \cap\{M_{f'} : f' \in \Delta'(M(A))\} \; (\because \Delta'(M(A)) \subseteq \Delta(M(A))) \\
&= \cap\{M_f : f \in \Delta(A)\} \quad (\because \Delta(A) \cong \Delta'(M(A))) \\
&= rad(A) = \{0\}.
\end{aligned}$$

Hence $rad\,(M(A)) = \{0\}$, and so $M(A)$ is semi-simple. ■

Definition 3.2.7. Let X and Y be normed spaces. For any $T \in BL(X,Y)$, the *adjoint* (or *conjugate or transpose*) of T is a mapping $T^+ \in B(Y^*, X^*)$ defined by

$$(T^+g)(x) = g(Tx), \; g \in Y^*, x \in X.$$

In particular, for any $T \in BL(A)$, the adjoint of T is a linear operator $T^+ \in BL(A^*) = BL(A^*, A^*)$ given by

$$(T^+g)(x) = g(Tx), \quad g \in A^*, x \in A.$$

Corollary 3.2.8. (Máté) *Let A be a semi-simple commutative Banach algebra and $T \in BL(A)$. Then the following are equivalent:*

(i) $T \in M(A)$

(ii) for each $f \in \Delta(A), \exists$ a constant $c(f)$ such that

$$T^+f = c(f)f.$$

Proof. $(i) \Rightarrow (ii)$. Suppose $T \in M(A)$, and let $f \in \Delta(A)$.Then, for any $x \in A$,

$$(T^+f)(x) = f(Tx) = f'(T).f(x) = c(f)f(x),$$

where $f' \in \Delta'(M(A))$ corresponds to f (as in the part (i)of Theorem 3.2.1) and $c(f) = f'(T)$. Hence $T^+f = c(f)f$.

$(\Leftarrow)$ Suppose $T^+f = c(f)f$ for each $f \in \Delta(A)$. Define $\phi : \Delta(A) \to \mathbb{C}$ by

$$\phi(f) = c(f), \quad f \in \Delta(A).$$

Then clearly $\phi \in C_b(\Delta(A))$ and $\phi\hat{A} \subseteq \hat{A}$ $(i.e.\phi \in m(A))$.Further , for each $x \in A$,

$$\begin{aligned} \widehat{Tx}\,(f) &= f(Tx) = (T^+ f)(x) \\ &= c(f)f(x) = \phi[f]\,\hat{x}\,[f]. \end{aligned}$$

Hence $\widehat{Tx} = \phi\,\hat{x}$ $(x \in A)$. Since A is semi-simple, by Corollary 3.1.5, $T \in M(A)$. ■

Corolloary 3.2.9. (Wang) *Let A be a semi-simple commutative supremum-norm algebra. Then $M(A)$ is a supremum norm-algebra.*

Proof. Since A is a supremum-norm algebra ,$\|x\| = \left\|\hat{x}\right\|_\infty$ for all $x \in A$, where

$$\left\|\hat{x}\right\|_\infty = \sup_{\substack{\phi \in A^* \\ \|\phi\| \le 1}} \left|\hat{x}\,(\phi)\right| = \sup_{\substack{\phi \in A^* \\ \|\phi\| \le 1}} |\phi(x)|\,.$$

Now, for any $T \in M(A)$, we define $\hat{T}$: $(M(A))^* \to \mathbb{K}$ by

$$\hat{T}\,(h) = h(T),\ h \in (M(A))^*.$$

Then $\hat{T}$ is linear and continuous, and so $\hat{T} \in (M(A))^{**}$.

To show that $M(A)$ is a supremum-norm algebra, we need to show that , for any $T \in M(A)$,

$$\|T\| = \left\|\hat{T}\right\|_\infty.$$

Now

$$\left\|\hat{T}\right\|_\infty = \sup_{\substack{h \in M(A)^* \\ \|h\| \le 1}} \left|\hat{T}\,(h)\right| = \sup_{\|h\| \le 1} |h(T)| \le \sup_{\|h\| \le 1} \|h\|\,\|T\| = \|T\|\,.$$

Also

$$\begin{aligned} \|T\| &= \sup_{\substack{x \in A \\ \|x\|=1}} \|Tx\| = \sup_{\left\|\hat{x}\right\|_\infty = 1} \left\|(\widehat{Tx})\right\|_\infty = \sup_{\left\|\hat{x}\right\|_\infty = 1} \left\|(T\widehat{Lx})\right\|_\infty \\ &= \sup_{\left\|\hat{x}\right\|_\infty = 1} \left\|(\hat{T}\hat{L}x)\right\| \le \sup_{\left\|\hat{x}\right\|_\infty = 1} \left\|(\hat{T})\right\|_\infty \left\|\hat{L}_x\right\|_\infty = \sup_{\|x\|=1} \left\|\hat{T}\right\|_\infty \|x\| = \left\|\hat{T}\right\|_\infty. \end{aligned}$$

Thus $\left\|\hat{T}\right\|_\infty = \|T\|$. So $M(A)$ is a supremum- norm algebra. ■

We next study $\Delta(A)$ and $\Delta(M(A))$ endowed with the hull-kernel topology τ_{hk}.

Theorem 3.2.10. (Wang) *Let A be a semi-simple commutative Banach lagebra, and let $\Delta(A)$ and $\Delta(M(A))$ be given the hull-kernel topology τ_{hk}. Then:*

(i) $\Delta(M(A) = H_\Delta(L_A) \cup \Delta'(M(A))$.

(ii) $H_{\Delta}(L_A)$ *is compact and* $\Delta'(M(A))$ *is open.*

(iii) $\Delta(A)$ *is homeomorphic to* $\Delta'(M(A))$ *under the map* $\theta : m \to m'$.

(iv) $\Delta'(M(A))$ *is* τ_{hk}*- dense in* $\Delta(M(A))$.

Proof. (i) This is proved in Theorem 3.2.4 .

(ii) By Theorem 3.2.4, $H_{\Delta}(L_A)$ is w^*-compact. Since $\tau_{hk} \leq w^*$, it follows that $H_{\Delta}(L_A)$ is τ_{hk}-compact.

Let $\{U\alpha : \alpha \in I\}$ be a τ_{hk}–open cover of $H_{\Delta}(L_A)$. Since $\tau_{hk} \leq w^*$, each $U\alpha$ is w* -open and so $\{U\alpha : \alpha \in I\}$ is a w^* -open cover of $H_{\Delta}(L_A)$. By Theorem 3.2.4, $H_{\Delta}(L_A)$ is w^*-compact and so $\exists$ $U_{\alpha 1}.....U_{\alpha n}$ such that $H_{\Delta}(L_A) \subseteq \underset{i=1}{\overset{n}{\cup}} U_{\alpha i}$.Hence $H_{\Delta}(L_A)$ is τ_{hk}–compact.

Next, $\Delta'(M(A))$ is open : Since $M(A)$ has an identity I, $\Delta(M(A))$ is τ_{hk}– compact. As proved above, $H_{\Delta}(L_A)$ is a τ_{hk}– compact subset of $\Delta(M(A))$. Therefore, its complement $\Delta'(M(A))$ is open.

(Note that if a subset E is τ_{hk}-compact, then it need not be τ_{hk}-closed since τ_{hk} need not be Hausdorff.)

(ii) In view of Theorem 3.2.1 and 3.2.4, the map $\theta : f \to f'$ of $\Delta(A)$ into $\Delta'(M(A))$ is one-one and onto. To prove that θ and $\theta^{-1}$are continuous, we first need to determine $\theta(\ hul\ (\ \ker\ [K]))$ for any $K \subseteq \Delta(A)$. Set $K' = \theta(K) = \{f' : f \in K\}$. Then, since $f'(L_y) = f(y)\ (y \in A)$,

$$\begin{aligned}
\ker(K) &= \bigcap_{f\in K} M_f = \{x \in A : x \in \bigcap_{f\in K} M_f\} \\
&= \{x \in A : L_x \in \bigcap_{f'\in K} M_{f'}\} = \{x \in A : L_x \in \ker(K')\}.
\end{aligned}$$

Further

$$ker(K) \subseteq M_f \Leftrightarrow \bigcap_{p\in K} p \subseteq M_f \Leftrightarrow \bigcap_{p'\in K'} M_{p'} \subseteq M_{f'} \Leftrightarrow \ker(K') \subseteq M_{f'}$$

Therefore

$$\theta\{hul(\ker[K])\} = \Delta'(M(A)) \cap hul(\ker[K']),$$

$$\begin{aligned}
\theta\{hul(\ker[K])\}' &= \{f \in \Delta(A) : \ker(K) \subseteq M_f\}' \\
&= \{f' \in \Delta(M(A)) : \ker(K) \subseteq M_{f'}\}
\end{aligned}$$

$$= \{f \in \Delta(A) : \{x \in A : L_x \in \ker(K')\} \subseteq M_{f'}\}$$

$$= \{f' \in \Delta(M(A)) : L_A \cap \ker(K') \subseteq M_{f'}\}$$

$$= \{f' \in \Delta(M(A)) : L_A \subseteq M_{f'}\} \cap \{f' \in \Delta(M(A)) : \ker(K') \subseteq M_{f'}\}$$

$$= \Delta'(M(A)) \cap hul(\ker[K'])$$

Now $\theta : \Delta(A) \to \Delta'(M(A))$ is τ_{hk} continuous, as follows. Let K' be a τ_{hk}-closed set in $\Delta'(M(A))$. Then

$$\begin{aligned}
\theta^{-1}(K') &= \theta^{-1}(\overline{K'}) = \theta^{-1}(hul(\ker[K'])) \\
&= \theta^{-1}[\Delta'(M(A)) \cap hul(\ker[K']) \\
&= \theta^{-1}(\theta[hul(\ker[K])] \\
&= hul(\ker[K]) \\
&= \overline{K} \quad (\text{the } \tau_{hk} - closure)
\end{aligned}$$

which is τ_{hk}-closed in $\Delta(A)$. Hence θ is continuous.

Next to show that $\theta^{-1} : \Delta'(M(A)) \to \Delta(A)$ is τ_{hk}−continuous. Let K be any τ_{hk}−closed set in $\Delta(A)$. Then

$$\begin{aligned}
(\theta^{-1})^{-1}(K) &= \theta(K) \quad (\because \theta \text{ is one-one and onto}) \\
&= \theta(\overline{K}) = \theta[hul(\ker[K])] \\
&= \Delta'(M(A)) \cap hul(\ker[K']) \\
&= \Delta'(M(A)) \cap \overline{K'},
\end{aligned}$$

which is τ_{hk}−closed in $\Delta'(M(A)$. Hence θ^{-1} is continuous. Thus θ is a homeomorphism.

(iv) $\Delta'(M(A))$ is τ_{hk}−dense in $\Delta(M(A))$: We show that $\overline{\Delta'(M(A))} = \Delta(M(A))$ *i.e.* hul(ker($\Delta'(M(A))) = \Delta(M(A))$, we first show that ker($\Delta'(M(A))) = \{0\}$. Now

$$L_x \in \ker(\Delta'(M(A))) \Rightarrow L_x \in M_{f'} \quad \text{for all } f' \in \Delta'(M(A)).$$

$$\Rightarrow L_x \in M_{f'} \quad \text{for all } f' \in \Delta'(M(A)).$$

$$\Rightarrow f'(L_x) = 0 \quad \text{for all } f' \in \Delta'(M(A)).$$

$$\Rightarrow f(x) = 0 \text{ for all } f' \in \Delta'(M(A)). [\because f'(L_x) = f(x)]$$

$$\Rightarrow x \in M_f \text{ for all } f \in \Delta(A). [\because \Delta(A) \cong \Delta'(M(A))].$$

$$\Rightarrow x \in \cap\{M_f : f \in \Delta(A)\} = rad\,(A) = \{0\}.$$

$$\Rightarrow x = 0.$$

$$\Rightarrow L_x = L_0 = 0$$

Hence $ker(\Delta'(M(A))) = \{0\}$.

Now

$$\begin{aligned} \overline{\Delta'(M(A))} &= hul(ker(\Delta'(M(A))) \\ &= hul(\{0\}) \\ &= \{f' \in \Delta(M(A)) : \{0\} \subseteq M_{f'}\} \\ &= \{\text{ all } f' \in \Delta(M(A))\} \\ &= \Delta(M(A)). \end{aligned}$$

Thus $\Delta'(M(A))$ is τ_{hk}-dense in $\Delta(M(A))$. ■

Note. (1) Parts (i),(ii),(iii) of the above theorem are also valid for A any faithful commutative algebra (not necessarily semi-simple).

(2) Part (iv) of the above theorem is not valid for semi-simple A with w^* topology on $\Delta(M(A))$ ($i.e.$ $\Delta'(M(A))$ need not be w^*-dense in $\Delta(M(A))$.

Definition 3.2.11. Let B be a commutative Banach algebra.

(1) B is said to be *regular* if $w^* = \tau_{hk}$ on $\Delta(B)$.

(2) B is called *self-adjoint* if, for each $x \in B$, $\exists$ an $x^* \in B$ such that

$$f(x^*) = \overline{f(x)} \quad \text{for all } f \in \Delta(A).$$

or $\hat{x}^*(f) = \overline{\hat{x}(f)}$ for all $f \in \Delta(A)$ $i.e.$ $\hat{x}^* = \overline{\hat{x}}$ on $\Delta(A)$.

The following example show that A may be both regular and self-adjoint but $M(A)$ may be neither.

Example 3.2.12. For any non-discrete locally compact abelian group G, let

$L_1(G) =$ the Banach algebra of all functions $f : G \to G$ which are absolutely integrable w.r.t. the Haar measure λ on G, norm

$$\|f\|_1 = \int_G |f(t)|\, d\lambda(t)$$

and usual convolution multiplication

$$(f * g)(t) = \int_G f(ts^{-1}) g(s) d\lambda(s),$$

and let $M(G) =$ the Banach algebra of all bounded regular Borel measures $\mu : B(G) \to \mathbb{C}$ with norm $\|\mu\| = |\mu|(G)$ and multiplication $(\mu * \gamma)(E) = \int_G \mu(Es^{-1}) d\gamma(s)$. It is well-known that $L_1(G)$ is a closed ideal in $M(G)$. Further, if $A = L_1(G)$, then

(1) $M(A) = M(G)$.

(2) Both A and $M(A)$ are commutative semi-simple Banach algebras.

(3) A is regular and self-adjoint.

(4) $\Delta(A) = \hat{G}$ which is not w^*-dense in $\Delta(M(G)) = \Delta(M(A))$.

But, by Theorem 3.2.10 $\Delta(A)$ is τ_{hk} -dense in $M(A)$.Hence $w^* \neq \tau_{hk}$ on $\Delta(M(A))$, and so $M(A)$ is not regular.

(5) $M(G)$ is self-adjoint $\Leftrightarrow$ G is discrete. Since G is not discrete, $M(A)$=M(G) is not self-adjoint. ■

Theorem 3.2.13. *Let A be a semi-simple commutative Banach algebra.*

(i) If $M(A)$ is regular, then A is regular.

(ii) If $M(A)$ is regular and A is self-adjoint, then $M(A)$ is self-adjoint.

Proof. (i) Suppose $M(A)$ is regular. Then $\tau_{hk} = w^*$ on $\Delta(M(A))$. By Theorem 3.2.4, $\Delta(A)$ is homeomorphic to a subspace $\Delta'(M(A))$ of $\Delta(M(A))$. Hence $\tau_{hk} = w^*$ on $\Delta(A)$. Therefore A is regular.

(ii) Suppose $M(A)$ is regular and A is self-adjoint. To show that $M(A)$ is self-adjoint, we need to show that, for any $T \in M(A)$, $\exists$ a $T^* \in M(A)$ *s.t.*

$$f'(T^*) = \overline{f'(T)} \quad \text{for all } f \in \Delta(M(A))$$
$$\hat{T^*}(f') = \overline{\hat{T}(f')} \quad \text{for all } f' \in \Delta(M(A)) \;\; i.e. \;\; \hat{T^*} = \overline{\hat{T}} .$$

Let $T \in M(A)$. Define $\phi^* : \Delta(A) \to \mathbb{K}$ by

$$\phi^*(f) = \overline{\hat{T}(f')}, \; f \in \Delta(A),$$

where $f' \in \Delta(M(A)$corresponds to $f \in \Delta(A)$ as in Theorem 3.2.4 since A is self-adjoint, for any $x \in A$, $\hat{x}(f) = \hat{x^*}(f)$, $f \in \Delta(A)$, for some $x^* \in A$. Then, for any $x \in A$ and f,

$$\begin{aligned} \phi^*(f)\,\hat{x}(f) &= \overline{T(f')}\hat{x}(f) = \overline{\hat{T}(f')(\overline{\hat{x}(f)})} \\ &= \overline{\hat{T}(f')\,\overline{\hat{x}(f)}} = \overline{\hat{T}(f')x^*(f)} \\ &= \overline{(T\hat{x^*})(f)} = [(\widehat{Tx^*})^*](f) = (\widehat{T^*x})(f) \end{aligned}$$

Hence $\phi^*\hat{x} = (\widehat{T^*x}) \in \hat{A}$ and so $\phi^*\hat{A} \subseteq \hat{A}$ .Clearly ϕ^* is bounded and continuous on $\Delta(A)$ and hence by the Corollary 3.1.3, ϕ^* defines a multiplier $T^* \in M(A)$ such that $\hat{T^*}(f') = \overline{\hat{T}(f')}$.

Since $M(A)$ is regular, $\Delta'(M(A))$ is w^*–dense in $\Delta(M(A)$ and so $\hat{T^*}(f') = \overline{\hat{T}(f')}$ for all $f' \in \Delta(M(A))$. Thus $M(A)$ is self-adjoint. ■

Note. The preceding example 3.2.12 with $A = L_1(G)$ shows that :

(1) If A is regular, then $M(A)$ need not be regular.

(2) If A is self-adjoint with $M(A)$ not regular, then $M(A)$ need not be self-adjoint.

Note. We have seen in Chapter 2 that if A is a faithful commutative Banach algebra, then A may be considered as a normed subalgebra of $M(A)$ under the embedding $\mu : A \to L_A \subseteq M(A)$. However, A (i.e. L_A) need not be norm-closed (or-complete) in $M(A)$.

The following result provides some information on the location of the norm-closure of A (*i.e.* of L_A) in $M(A)$.

Theorem 3.2.14. (Wang) *Let A be a faithful commutative Banch algebra. Then the norm closure of L_A in $M(A)$ is contained in $ker(H(L_A))$.*

Proof. Let T belong to the norm-closure of L_A. To show that $T \in \ker(H(L_A))$, we need to show that, for any $f' \in H(L_A) = \Delta(M(A)) \setminus \Delta'(M(A))$, $f'(T) = 0$. Let $\varepsilon > 0$. Since $T \in \tau_A$,$\exists\, y \in A$ such that

$$\|T - L_y\| < \varepsilon.$$

By Corollary 3.2.5, the Gelfand transform $\widehat{L_y}$ of L_A vanishes on $H(L_A)$. Hence, for each $f' \in H(L_A)$, $\widehat{L_A}$ $(f') = 0$ and so

$$\begin{aligned} |f'(T)| &= \left|\widehat{T}(f')\right| = \left|\widehat{T}(f') - \widehat{L_y}(f')\right| = \left|(T \widehat{-} L_y)(f')\right| \\ &= |f'(T - L_y)| \le \|f'\| \, \|T - L_y\| < 1.\varepsilon = \varepsilon. \end{aligned}$$

Since ε is arbitrary, $f'(T) = 0$ for all $f' \in H(L_A)$. Hence $T \in \ker(H(L_A))$, and so $\tau_A \subseteq \ker(H(L_A))$. ■

We have seen that when A is a supremum norm algebra, by Theorem 3.1.4, the correspondence between $M(A)$ and $m(A)$ is an isometry and, by Corollary 3.2.9, $M(A)$ is itself a supremum norm algebra. We now present some more results in this direction.

Theorem 3.2.15. (Wang) *Let A be a semi-simple commutative supremum-norm algebra and suppose L_A is s-dense in $M(A)$. If ϕ is a bounded function on $\Delta(A)$ such that $\phi \widehat{A} \subseteq \widehat{M(A)}$, then $\phi \in \widehat{M(A)}$; that is, ϕ defines a multiplier for $M(A)$.*

Proof. Since L_A is s-dense in $M(A)$, A contains an approximate identity $\{e_\alpha\}$(see Section 2.3). Since $\phi \widehat{A} \subseteq \widehat{M(A)}$, there exists $\{T_\alpha\} \subseteq M(A)$ such that

$$\phi \, \widehat{e_\alpha} = \widehat{T_\alpha};$$

hence

$$\phi \, (\widehat{L_{e_\alpha} x}) = \phi \, (\widehat{e_\alpha x}) = \widehat{(T_\alpha x)} \quad (x \in A)$$

Moreover, for each $x \in A$,

$$\begin{aligned} \|T_\alpha x - T_\beta x\| &= \left\|(T_\alpha x - T_\beta x)^{\wedge}\right\|_\infty = \left\|(\widehat{T_\alpha x}) - (\widehat{T_\beta x})\right\|_\infty \\ &= \left\|\phi \, (\widehat{e_\alpha x}) - \phi \, (\widehat{e_\beta x})\right\|_\infty = \left\|\phi \, [(e_\alpha x)^{\wedge} - (e_\beta x)^{\wedge}]\right\|_\infty \\ &\le \|\phi\|_\infty \left\|(e_\alpha x - e_\beta x)^{\wedge}\right\|_\infty = \|\phi\|_\infty \, \|e_\alpha x - e_\beta x\| \\ &\le \|\phi\|_\infty \left(\|e_\alpha x - x\| + \|x - e_\beta x\|\right) \end{aligned}$$

Since $\|e_\alpha x - x\| \to 0$ and $\|x - e_\beta x\| \to 0$, it follows that $\{T_\alpha x\}$ is a Cauchy net in A. Hence $\{T_\alpha\}$ is a s-Cauchy net in $M(A)$. Since $(M(A), s)$ is complete by a Theorem in Chapter 2,

$$T_\alpha \xrightarrow{s} T \in M(A).$$

Hence, for any $x \in A$,

$$\begin{aligned} \phi \hat{x} &= \phi(\lim_\alpha e_\alpha x \hat{)} = \lim_\alpha \phi\, (e_\alpha \hat{x}) \\ &= \lim_\alpha (\hat{T_\alpha x})) = (\hat{Tx}) \,. \end{aligned}$$

Therefore $\phi = \hat{T} \in \hat{M(A)}$, and so ϕ defines a multiplier $T \in M(A)$. ■

Theorme 3.2.16. *Let A be a semi-simple commutative supremum-norm algebra. Then the closure of $\partial(\Delta'(M(A))$ in $\Delta(M(A))$ is equal to $\partial(\Delta(M(A))$.*

Proof. By Corollary 3.2.5, for each $x \in A$,

$$\hat{L_x} = 0 \qquad \text{outside } (\Delta'(M(A));$$

i.e.

$$\hat{L_x} = 0 \text{ on } \Delta(M(A) \setminus \Delta'(M(A);$$

i.e.

$$\hat{L_x}\,(p) = 0 \quad \text{for all } p \in \Delta(M(A) \setminus \Delta'(M(A).$$

Since $\hat{L_x}\ (f') = f'(L_x) = f(x) = \hat{x}\ (f)$, it follows that $\partial(\Delta'(M(A)) \subseteq \partial(\Delta(M(A))$. [In fact, $f' \in \partial(\Delta'(M(A)) \Leftrightarrow$ for each neighbourhood U of f', $\exists\ \hat{L_y} \in \hat{M(A)}$ such that

$$\begin{aligned} &\sup_{f' \in \Delta'(M(A)) \setminus U} \left|\hat{L_y}\,(f')\right| < \sup_{f \in U} \left|\hat{L_y}\,(f')\right| \\ \Rightarrow &\sup_{f' \in \Delta(M(A)) \setminus U} \left|\hat{L_y}\,(f')\right| < \sup_{f' \in U} \left|\hat{L_y}\,(f')\right| \\ &\Rightarrow f' \in \partial(M(A)).] \end{aligned}$$

Hence $\overline{\partial\Delta'(M(A)))} \subseteq \overline{\partial\Delta(M(A))} = \partial(\Delta M(A))$ (since $\partial\Delta(M(A))$ is closed).

Next to show that $\partial\Delta(M(A)) \subseteq \partial\Delta'(M(A))$, we first show that, for any $T \in M(A))$,

$$\left\|\hat{T}\right\|_\infty \leq \sup_{f' \in \partial\Delta'(M(A))} \left|\hat{T}\,(f')\right|$$

$$\begin{aligned} \left\|\hat{T}\right\|_\infty &= \|T\| = \sup_{\substack{x \in A \\ \|x\|=1}} \|Tx\| = \sup_{\|x\|=1} \left\|(\hat{Tx})\right\|_\infty \\ &= \sup_{\|x\|=1} \sup_{f \in \partial\Delta(A)} \left|(\hat{Tx})\,(f)\right| \\ &= \sup_{\|x\|=1} \sup_{f \in \partial\Delta(A)} \left|\hat{T}\,(f')\,\hat{L_x}\,(f')\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|x\|=1} \sup_{f\in\partial\Delta(A)} \left|\overset{\wedge}{L_x}(f')\right| \cdot \sup_{m\in\partial\Delta(A)} \left|\hat{T}(f')\right| \\
&= \sup_{\|x\|=1} \sup_{f\in\partial\Delta(A)} \left|\hat{x}(f)\right| \sup_{f\in\partial\Delta(A)} \left|\hat{T}(f')\right| \\
&= \sup_{\|x\|=1} \left\|\hat{x}\right\|_\infty \cdot \sup_{f'\in\partial\Delta'(M(A))} \left|\hat{T}(f')\right| \\
&= \sup_{\|x\|=1} \|x\|_\infty \cdot \sup_{f'\in\partial\Delta'(M(A))} \left|\hat{T}(f')\right| = \sup_{f'\in\partial\Delta'(M(A))} \left|\hat{T}(f')\right|
\end{aligned}
$$

Consequently, $\partial\Delta(M(A)) \subseteq \partial\Delta'(M(A))$ (since $\partial\Delta(M(A))$ is the minimal closed maximizing modulus set in $\Delta(M(A))$).Thus $\partial\Delta(M(A)) = \overline{\partial\Delta'(M(A))}$.

Corollory 3.2.17. *Let A be a semi-simple supremum-norm algebra. If $T \in M(A)$ and $\hat{T}= 0$ on $\partial\Delta'(M(A))$, then T is a zero multiplier.*

Proof. Since $\hat{T}$ is continuous on $\Delta(M(A))$ and $\hat{T}= 0$ on $\partial\Delta'(M(A))$, by a result, $\hat{T}= 0$ on $\overline{\partial\Delta'(M(A))}$. Then , by above theorem, $\hat{T}= 0$ on $\partial\Delta(M(A))$. Now, using the definition of $\partial\Delta(M(A))$

$$
\begin{aligned}
\|T\| &= \left\|\hat{T}\right\|_\infty = \sup_{f'\in\Delta M(A)} \left|\hat{T}(f')\right| \\
&= \sup_{f'\in\partial\Delta M(A)} \left|\hat{T}(f')\right| = 0.
\end{aligned}
$$

Thus $T = 0$. ■

Corollary 3.2.18. *Let A be a semi-simple commutative supremum-norm algebra with $\Delta(M(A)) = \Delta'(M(A))$ then*

(i) $\partial\Delta'(M(A))$ is closed in $\Delta(M(A))$.

(ii) $\partial\Delta'(M(A)) = \partial\Delta(M(A))$.

Notation. Let A be a commutative Banach algebra. If the *Bishop boundary* $\rho\Delta(A)$ exists, we denote by $\rho\Delta'(M(A))$ its imag under the homeomorphism $m \to m'$ which maps $\Delta(A)$ into $\Delta(M(A))$.

Theorem 3.2.19. *Let A be a semi-simple commutative supremum-norm algebra. If $\rho\Delta(A)$ sxists, then*

$$\rho\Delta'(M(A)) = \rho\Delta(M(A))$$

Proof. As in the proof of Theorem 3.2.16, $\rho\Delta'(M(A)) \subseteq \rho\Delta(M(A))$.Since A is a supremum norm algebra, by Corollary 3.2.9, $M(A)$ is also a supremum-norm algebra. Again (as in Theorem 3.2.16

), for any $T \in M(A)$,

$$\begin{aligned}\left\|\hat{T}\right\|_{\infty} &= \|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \left\|\hat{Tx}\right\|_{\infty} = \sup_{\|x\|=1} \sup_{f\in\Delta(A)} \left|(\hat{Tx})\,(f)\right| \\ &= \sup_{\|x\|=1} \sup_{f\in\rho\Delta(A)} \left|(\hat{Tx})\,(f)\right| \\ &\leq \sup_{\|x\|=1} \sup_{f\in\rho\Delta(A)} \left|\hat{T}\,(f')\,\hat{L_x}\,(f')\right| \\ &\leq \sup_{\|x\|=1} \sup_{f\in\rho\Delta(A)} \left|\hat{x}\,(f)\right| \sup_{m\in\rho\Delta(A)} \left|\hat{T}\,(f')\right| \\ &= \sup_{\|x\|=1} \left\|\hat{x}\right\|_{\infty} \cdot \sup_{f'\in\rho\Delta'(A)} \left|\hat{T}\,(f')\right| = \sup_{f'\in\rho\Delta'(A)} \left|\hat{T}\,(f')\right|.\end{aligned}$$

Thus $\rho\Delta(M(A)) \subseteq \rho\Delta'(M(A))$. ■

In the following, we give an alternate proof of an earlier result (Theorem 3.1.4) stated again as:

Theorem 3.2.20. *Let A be a semi-simple commutative supremum-norm algebra.Then the correspondence between $M(A)$ and $m(A)$ given by $T \to \phi$, where*

$$(\hat{Tx})\,(f) = \phi(f)\,\hat{x}\,(f) \quad (f \in \Delta(A)\,)$$

is an isometric isomorphism (Hence $m(A)$ is a semi-simple Banach algebra of bounded continuous functions on $\Delta(A)$.)

Proof. By Theoerem 3.1.1, $\|\phi\|_{\infty} \leq \|T\|$. On the other hand, since $M(A)$ a supremum norm algebra, the proof of Theorem 3.1.4 shows that

$$\begin{aligned}\|T\| &= \left\|\hat{T}\right\|_{\infty} = \sup_{f'\in\partial\Delta'(M(A))} \left|\hat{T}\,(f')\right| = \sup_{f\in\partial\Delta(A)} \left|\frac{(\hat{Tx})\,(f)}{\hat{x}\,(f)}\right| \\ &= \sup_{f\in\partial\Delta(A)} |\phi(f)| \leq \sup_{f\in\Delta(A)} |\phi(f)| = \|\phi\|_{\infty}\end{aligned}$$

Thus $\|T\| = \|\phi\|_{\infty}$. ■

3.3. Isometric Multipliers

In this section we present some results on isometric multipliers which were introduced by Wermer [27] and Birtel [2] and further unified by Larsen [19]. It is shown that the set of isometric multipliers forms a Hausdorff topological group with respect to the strict topology and is compact under some additional assumptions (although. it need not be a vector space).

Definition 3.3.1. Let A be a normed algebra. A linear mapping $T : A \to A$ is said to be *isometric* if

$$\|Tx\| = \|x\| \quad \text{for all } x \in A$$

In this case, clearly T is one-one and $\|T\| = 1$.

Let $\vartheta(A) =$ the set of all isometric onto multipliers $T \in M(A)$.

Note that if $S, T \in \vartheta(A)$ and $\alpha \in \mathbb{K}$, then $S + T$, αT need not be isometric and so $\vartheta(A)$ need not be a vector subspace of $M(A)$. However, $\vartheta(A)$ is a topological group under multiplication, as follows.

Lemma 3.3.2. *Let A be a semi-simple commutative Banach algebra. Then*

(i) $\vartheta(A)$ is a commutative group with operation composition $S \circ T$ given by

$$(S \circ T)(x) = S(T(x)), \quad x \in A.$$

(ii) $(\vartheta(A), s)$ is a Hausdorff topological group.

Proof. (i) (G_1) Let $S, T \in \vartheta(A)$. We need to show that $S \circ T$ is isometric and onto. For any $x \in A$,

$$\|(S \circ T)x\| = \|S(Tx)\| = \|Tx\|$$

Hence $S \circ T$ is isometric. To show that $S \circ T$ is onto, let $z \in A$. Since S is onto $\exists\ y \in A$ such that $S(y) = z$. Since T is onto ,$\exists\, x \in A$ such that $T(x) = y$. Hence

$$(S \circ T)(x) = S(Tx) = Sy = z,$$

and so $S \circ T$ is onto. Hence $S \circ T \in \vartheta(A)$.

(G_2) Since $\vartheta(A) \subseteq M(A)$ and $M(A)$ is an algebra, so the associtive low holds in $\vartheta(A)$.

(G_3) The identitiy map $I : A \to A$ is clearly one-one. For any $x \in A$, $\|I(x)\| = \|x\|$, and so I is an isometry. Hence $I \in \vartheta(A)$.

(G_4) Let $T \in M(A)$. Since T is one-one and onto, $T^{-1} : A \to A$ exists and $T^{-1} \in M(A)$. Clearly T^{-1} is onto. To show that T^{-1} is an isometry, let $y \in A$ and let $x = T^{-1}(y)$. Then $T(x) = y$. Hence, since $\|x\| = \|Tx\|$,

$$\|T^{-1}(y)\| = \|x\| = \|Tx\| = \|y\|$$

This proves that T^{-1} is an isometry and hence $T^{-1} \in M(A)$.

(G_5) Since $\vartheta(A) \subseteq M(A)$ and $M(A)$ is commutative , $\vartheta(A)$ is commutative.

Thus $\vartheta(A)$ is a commutative group .

(ii) (TG_1) We show that the multiplication operation $(S,T) \to S \circ T$ is jointly continuous at $(S,T) = (I,I)$ in $\vartheta(A) \times \vartheta(A)$. Let $\{S_\alpha\}$ and $\{T_\beta\}$ be nets in $\vartheta(A)$ such that $S_\alpha \xrightarrow{s} I$ and $T_\beta \xrightarrow{s} I$. We need to show that $S_\alpha \circ T_\beta \xrightarrow{s} I \circ I = I$. For any $x \in A$,

$$\begin{aligned}
\|(S_\alpha \circ T_\beta)(x) - I(x)\| &\leq \|S_\alpha(T_\beta x) - S_\alpha(x)\| + \|S_\alpha x - I(x)\| \\
&= \|S_\alpha(T_\beta x) - I(x)\| + \|S_\alpha x - I(x)\| \\
&= \|T_\beta x - I(x)\| + \|S_\alpha x - I(x)\| \to 0.
\end{aligned}$$

Hence $S_\alpha \circ T_\beta \xrightarrow{s} I$ and so the muliplication is jointly continuous in $\vartheta(A)$.

(TG_2) We show that the inversion $T \to T^{-1}$ is continuous at $T = I$. Let $\{T_\alpha\}$ be a net in $\vartheta(A)$ such that $T_\alpha \xrightarrow{s} I$. Then, for any $x \in A$,

$$\begin{aligned}
\|T_\alpha^{-1}x - I^{-1}x\| &= \|T_\alpha^{-1}x - x\| \\
&= \|T_\alpha(T_\alpha^{-1}x) - x\| \\
&= \|T_\alpha(T_\alpha^{-1}x) - T_\alpha x\| \\
&= \|x - T_\alpha x\| \to 0.
\end{aligned}$$

Hence $T_\alpha^{-1} \xrightarrow{s} I^{-1} = I$, and so the inversion is continuous.

Thus $(\vartheta(A), s)$ is a topological group.

Finally $(\vartheta(A), s)$ is Hausdorff, as follows. Let $T_1, T_2 \in \vartheta(A)$ with $T_1 \neq T_2$. Then $T_1 x \neq T_2 x$ for some $x(\neq 0)$ in A . Let $\varepsilon = \|T_1 x - T_2 x\|$, and let

$$U = \{T \in \vartheta(A) : \|Tx - T_1 x\| < \frac{\varepsilon}{4}\}, V = \{T \in \vartheta(A) : \|Tx - T_2 x\| < \frac{\varepsilon}{4}\}$$

Then U and V are s-open neighbourhoods of $T_1$and T_2,respectively. Suppose $U \cap V \neq \phi$, and let $T \in U \cap V$. Then

$$\varepsilon = \|T_1 x - T_2 x\| \leq \|T_1 x - Tx\| + \|Tx - T_2 x\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon/2,$$

a contradiction. Therefore $U \cap V = \phi$. Hence $(\vartheta(A), s)$ is Hausdorff. (Alternatively, let $T\ (\neq 0) \in M(A)$. Then $Tx \neq 0$ for some $x(\neq 0)$ in A. Therefore $P_x(T) = \|Tx\| \neq 0$, and so $(M(A), s)$ is Hausdorff. Hence $(\vartheta(A), s)$, being a subspace of $(M(A), s)$, is Hausdorff.) ■

We next consider verious assumption on the topological group $(\vartheta(A), s)$ so that it is compact.

Notation. For each $x \in A$, define

$$O_x = \{Tx : T \in \vartheta(A)\}.$$

Lemma 3.3.3. *Let A be a semi-simple commutative Banach algebra . Suppose O_x has compact norm-closure in A for each $x \in A$. Then $(\vartheta(A), s)$ is compact.*

Proof. Let $Q = \prod_{x \in A} \overline{O}_x$, where $\overline{O}_x$ is the norm-closure of O_x in X. Since each $\overline{O}_x$ is compact, by the Tychonoff theorem, Q is compact. Let $\overline{\wedge}_x$ be the projection of Q onto $\overline{O}_x$. Define $\tau : \vartheta(A) \to Q$ by

$$\overline{\wedge}_x[\tau(T)] = Tx \quad (T \in \vartheta(A), x \in A)$$

and $w : \vartheta(A) \to Q$ by

$$\overline{\wedge}_x[w(T)] = T^{-1}x \quad (T \in \vartheta(A), x \in A)$$

To show that $\vartheta(A)$ is compact, we take any net $\{T_\alpha\}$ in $\vartheta(A)$ and show that it has a convergent subset. Now $\{\tau(T_\alpha)\}$ and $\{w(T_\alpha)\}$ are nets in the compact space Q and hence have convergent subnets. Without loss of generality we may assume that $\{T_\beta\}$ is a subset of $\{T_\alpha\}$such that

$$\tau(T_\beta) \to t \in Q \text{ and } w(T_\beta) \to t' \in Q$$

Define $T, T' : A \to A$ by

$$Tx = \overline{\wedge}_x(t) = \lim_\beta T_\beta x, \qquad 0.5$$

$$T'x \;=\; \overline{\wedge}_x(t') = \lim_\beta T_\beta^{-1}x, \quad (x \in A).$$

Then T and T' are linear since T_β and T_β^{-1} are linear. Further, since T_β and T_β^{-1} are isometric

$$\|Tx\| = \lim_\beta \|T_\beta x\| = \lim_\beta \|x\| = \|x\|$$
$$\left\|T'x\right\| = \lim_\beta \left\|T_\beta^{-1}x\right\| = \lim_\beta \|x\| = \|x\|$$

and so T and T' are isometric.

To see that T and T' are multipliers on A, let $x, y \in A$. Then

$$T(xy) = \lim_\beta T_\beta(xy) = \lim_\beta x(T_\beta y) = x(Ty),$$
$$T'(xy) = \lim_\beta T_\beta^{-1}(xy) = \lim_\beta x(T_\beta^{-1}y) = x(T'y),$$

and so $T, T^{-1} \in M_r(A) = M(A)$.

Since T is one-one, to show that T is onto, we need to show T^{-1} exists and $T^{-1} \in M(A)$. For any $x \in A$, (1) gives

$$\begin{aligned} \|T(T'x) - I(x)\| &\leq \|T(T'x) - T_\beta(T'x)\| + \left\|T_\beta[T'x - T_\beta^{-1}x]\right\| \\ &= \|T(T'x) - T_\beta(T'x)\| + \left\|T'x - T_\beta^{-1}x\right\| \\ &\to 0 + 0 = 0 \end{aligned}$$

Hence $T(T') = I = T'(T)$ and so $T^{-1} = T'$ exists and belongs to $M(A)$. Therefore T is onto (In fact $T(T'x) = x$ also shows that T is onto). Thus $T \in \vartheta(A)$. Clearly , by (1), $T_\beta \xrightarrow{s} T$. This proves that $(\vartheta(A), s)$ is compact. ■

Recall that if X and Y are normed spaces, then a linear mapping $T : X \to Y$ is called *compact* if T maps bounded subsets of X into relatively compact subsets of Y.

Corollary 3.3.4. *Let A be a semi-simple commutative Banach algebra with a minimal approximate identity. If $L_x : A \to A$ is a compact operator for each $x \in A$, then $(\vartheta(A), s)$ is compact.*

Proof. Let $\{e_\alpha\}$ be a minimal approximate identity in A with $\|e_\alpha\| = 1$. In view of the above theorem , we need to show that , for each $x \in A$, $\overline{O_x}$ is compact. Let $x \in A$, and let $T \in M(A)$. Since A has a minimal approximate identity,

$$\|L_a\| = \|a\| \quad (a \in A)$$

and so

$$\left\|L_{Te_\alpha}\right\| = \|Te_\alpha\| = \|e_\alpha\| = 1.$$

Further ,

$$L_{Te_\alpha}(x) = (Te_\alpha)x = e_\alpha(Tx) \to Tx$$

and so

$$O_x = \{Tx : T \in \vartheta(A)\} \subseteq \{L_{Te_\alpha}(x) : \alpha \in J,\ T \in \vartheta(A)\} \text{-----}(1)$$

Now

$$\begin{aligned} \{L_{Te_\alpha}(x)\ : \alpha \in J,\ T \in \vartheta(A)\} &\subseteq \{L_y(x) : y \in A, \|y\| \leq 1\} \\ &= \{L_x(y) : y \in A, \|y\| \leq 1\} \\ &= L_x(B_1(0)).\text{-----}(2) \end{aligned}$$

By (1) and (2),

$$O_x \subseteq \overline{L_x(B_1(0))}\ .$$

Since L_x is a compact operator, $\overline{L_x(B_1(0))}$ is compact and so $\overline{O_x}$ is also compact. ■

Before we give another criteria for the compactness of $(\vartheta(A), s)$, we make the following definition.

An element $x \in A$ is said to be *almost invariant* if O_x spans a finite dimensional subspace of A. Let $D(A)$ be the set of all almost invariant elements in A. Then, $D(A)$ is a subalgebra of A.[Let $x, y \in D(A)$ and $\alpha, \beta \in \mathbb{K}$. Since span($O_x$) and span($O_y$) are finite-dimensional, it follows that span ($O_{\alpha x+\beta y}$) and span (O_{xy}) are also finite-dimensional. Hence $\alpha x + \beta y, xy \in D(A)$, and so $D(A)$ is a subalgebra of A.]

Corollary 3.3.5. *Let A be a semi-simple commutative Banach algebra. If $D(A)$ is norm-dense in A, then $(\vartheta(A), s)$ is compact.*

Proof. If $x \in D(A)$, then, since $O_x$spans a finite-dimensional space X_n(say), it follows that $\overline{O_x}$ is a norm-closed and bounded subset of X_n and hence norm-compact by the Heine-Borel theorem. We need to show that, for each $x \in A$, $\overline{O_x}$ is norm-compact.

Let $x \in A$. Since $D(A)$ is norm-dense in A , $\exists\ \{x_n\} \subseteq D(A)$ such that

$$\|x_n - x\| < \frac{1}{2^{n+1}}\ , \qquad n = 1, 2, 3, \ldots\ldots\text{--------}(1)$$

Set $Q = \prod_{n=1}^{\infty} \overline{O_{x_n}}$, where each $\overline{O_{x_n}}$ is compact (since $x_n \in D(A)$). Let $\overline{\wedge}_{x_n}$ be the projection of Q onto $\overline{O_{x_n}}$. Define $\tau : \vartheta(A) \to Q$ by $\overline{\wedge}_{x_n}[\tau(T)] = Tx_n, T \in \vartheta(A)$.

Now to show that $\bar{O}_x$ is norm-compact, it suffices to prove that every net $\{T_\alpha x\} \subseteq O_x$ has a subnet which convergent in $\bar{O}_x$. [For if this holds but $\bar{O}_x$ is not norm-compact ,then there exists a net in $\bar{O}_x$ which has no convergent subnet. Since O_x is dense in $\bar{O}_x$, we can construct a net in O_x which also has no convergent subnet in $\bar{O}_x$ contrary to our assumption.]

Let $\{T_\alpha x\}$ be a net in O_x. Since $\{T_\alpha\}$ is a net in $\vartheta(A)$, as in the proof of Lemma 3.3.3, $\{\tau(T_\alpha)\}$ is a net in the compact apace Q and hence has a convergent subnet $\{\tau(T_\beta)\}$ with $\tau(T_\beta) \to t \in Q$.

$$\begin{aligned}
\left\|\overline{\wedge}_{x_{n+1}}(t) - \overline{\wedge}_{x_n}(t)\right\| &= \left\|\lim_\beta T_\beta x_{n+1} - \lim_\beta T_\beta x_n\right\| \\
&= \lim_\beta \left\|T_\beta(x_{n+1} - x_n)\right\| \\
&= \lim_\beta \left\|x_{n+1} - x_n\right\| \\
&\leq \left\|x_{n+1} - x\right\| + \left\|x - x_n\right\| \\
&< \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} .
\end{aligned}$$

This implies that $\{\overline{\wedge}_{x_n}(t)\}$ is Cauchy in A and hence norm-convergent to some $y \in A$. Then

$$\begin{aligned}
\|T_\beta x - y\| &\leq \|T_\beta x - T_\beta x_n\| + \|T_\beta x_n - \overline{\wedge}_{x_n}(t)\| + \|\overline{\wedge}(t) - y\| \\
&= \|T_\beta(x - x_n)\| + \|\overline{\wedge}_{x_n}[\tau(T_\beta)] - \overline{\wedge}_{x_n}(t)\| + \|\overline{\wedge}_{x_n}(t) - y\| \\
&= \|x - x_n\| + \|\overline{\wedge}_{x_n}[\tau(T_\beta)] - \overline{\wedge}_{x_n}(t)\| + \|\overline{\wedge}_{x_n}(t) - y\| \\
&< \frac{1}{2^{n+1}} + \|\overline{\wedge}_{x_n}[\tau(T_\beta)] - \overline{\wedge}_{x_n}(t)\| + \|\overline{\wedge}_{x_n}(t) - y\| \\
&\to 0. \qquad \text{(since } \tau(T_\beta) \to t \text{ and } \overline{\wedge}_{x_n}(t) \to y\text{)}
\end{aligned}$$

Hence $\lim_\beta T_\beta x = y$, where $y \in \bar{O}_x$ (since $\{T_\beta x\} \subseteq O_x$). Therefore $\bar{O}_x$ is norm-compact for each $x \in A$ and thus, by Lemma 3.3.3, $(\vartheta(A), s)$ is compact. ■

CHAPTER 4

MULTIPLIERS ON COMMUTATIVE FRÉCHET ALGEBRAS

In this chapter, we study multipliers in the more general setting of commutative Fréchet algebras. In particular, we present extensions of some results of Chaper 3 from Banach algebras to topological algebras, due to Husain [12] and Khan-Mohammed-Thaheem [18], in refined and slightly improved form.

4.1. Multipliers on Topological Algebras

In this section we first establish the continuity and linearity of multipliers on topological algebras by using two important results: the Cohen factorization theorem and the closed graph theorem.

The following theorem from [18] extends some results in [16] and [6] to a class of topological algebras.

Theorem 4.1.1. *Let A be a fundamental F-algebra with a left (resp. right) uniformly bounded approximate identity. Then any left (resp. right) multiplier T on A is linear and continuous.*

Proof. Let T be a left multiplier on A. Considering A as a left A-module over itself, by the Cohen factorization theorem (see Section 1.4), given any sequence $\{a_n\} \subseteq A$ with $a_n \longrightarrow 0$, there exist $b \in A$ and $\{c_n\} \subseteq A$ with $c_n \longrightarrow 0$ such that $a_n = bc_n$. To show that T is linear, let $a_1, a_2 \in A$ and $\alpha, \beta \in \mathbb{K}$. Taking $\{a_n\} = \{a_1, a_2, 0, 0, ...\}$, there exist $b, c_1, c_2 \in A$ such that $a_1 = bc_1, a_2 = bc_2$. So

$$\begin{aligned} T(\alpha a_1 + \beta a_2) &= T(b(\alpha c_1 + \beta c_2)) = T(b)(\alpha c_1 + \beta c_2) \\ &= \alpha(bc_1) + \beta T(bc_2) = \alpha T(a_1) + \beta(a_2); \end{aligned}$$

hence T is linear. To show that T is continuous, it suffices to show that if $\{a_n\} \subseteq A$ with $a_n \longrightarrow 0$, then $T(a_n) \longrightarrow 0$. Using again the factorization theorem, we can write $a_n = bc_n$, where $b \in A$ and $\{c_n\} \subseteq A$ with $c_n \longrightarrow 0$. Then

$$T(a_n) = T(bc_n) = T(b)c_n \longrightarrow T(b)0 = 0.$$

(by the separate continuity of multiplication). Thus T is continuous.

In the case of T a right multiplier, we can apply the right hand version of the factorization theorem and obtain, as above, the linearity and continuity of T. ■

Theorem 4.1.2. *Let A be a faithful F-algebra. Then any multiplier T on A is linear and continuous.*

Proof. Let T be a multiplier on A. Since A is faithful, we have already seen in Chapter 2 that T is linear. For the continuity of T, we first show that T has closed graph. Let $\{a\alpha\}$ be a net in A with $a\alpha \longrightarrow a \in A$ and $T(a\alpha) \longrightarrow x \in A$. By the separate continuity of multiplication, for any $b \in A$, $T(b)a\alpha \longrightarrow T(b)a$ and $bT(a\alpha) \longrightarrow bx$; hence

$$bT(a) = T(b)a = \lim_{\alpha} T(b)a\alpha = \lim_{\alpha} bT(a\alpha) = bx.$$

Since A is faithful, $T(a) = x$; so T has closed graph. Since A is an F-space, by the closed graph theorem, T is continuous. ■

Now, we define some topologies on algebras of multipliers.

Definition 4.1.3. Let A be a Hausdorff topological algebra with $\mathbb{W}$ a base of neighbourhoods of 0. Let now $M_l(A)$ (resp. $M_r(A)$, $M(A)$) denote the set of all continuous and linear left multipliers (resp. right multipliers, multipliers) on A. The *uniform operator topology* $u = u_A$ (resp. the *strong operator topology*) on $M_l(A)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(D, G) = \{T \in M_l(A) : T(D) \subseteq G\},$$

where D is a bounded (respectively, finite subset) of A and $G \in \mathbb{W}$. The topology s is sometimes called the *strict topology* in the literature and denoted by $\beta = \beta_A$. Clearly, $s \leq u$. It is easy to see that $M_l(A)$ endowed with each of u and s is a semitopological algebra. (The above topologies can also be defined on $M_r(A)$ and $M(A)$, but we restrict ourselves to $M_l(A)$).

Next, we obtain analogues of some results of [18] to algebras of continuous left and right multipliers.

Theorem 4.1.4. *Suppose that (A, q) is an F-algebra with a uniformly bounded left approximate identity $\{e\lambda : \lambda \in I\}$. Then:*

(a) $(M_l(A), u)$ is complete.

(b) If $q(e\lambda) \leq 1$ for all $\lambda \in I$, then $\mu_l(A)$ is a u-closed left ideal in $M_l(A)$.

Proof. (a) Let $\{T\alpha : \alpha \in J\}$ be a Cauchy net in $(M_l(A), u)$. Then it easily follows that, for each $a \in A$, $\{T\alpha(a)\}$ is a Cauchy net in A. Consequently, the mappings $T : A \longrightarrow A$, given by $T(a) = \lim\limits_{\alpha} T\alpha(a)$ $(a \in A)$, is well-defined. Further, for any $a, b \in A$,

$$T(ab) = \lim_{\alpha} T\alpha(ab) = \lim_{\alpha} T\alpha(a)b = T(a)b;$$

hence $T \in M_l(A)$. We now show that $T\alpha \xrightarrow{u} T$. Let D be a bounded subset of A and take closed $G \in \mathbb{W}$. There exists an index α_o such that

$$T\alpha(a) - T\gamma(a) \in G \text{ for all } a \in D \text{ and } \alpha, \gamma \geq \alpha_o.$$

Since G is closed, fixing $\alpha \geq \alpha_o$ and taking $\lim\limits_{\gamma}$, we have

$$T\alpha(a) - T(a) \in G \text{ for all } a \in D.$$

Hence, for any $\alpha \geq \alpha_o$,

$$T\alpha - T \in N(D, G).$$

Thus $(M_l(A), u)$ is complete.

(b) We have already seen that $\mu_l(A)$ is a two-sided ideal in $M_l(A)$. To show that $\mu_l(A)$ is u-closed in $M_l(A)$, let $T \in M_l(A)$ with $T \in \overline{\mu_l(A)}^u$. There exists a net $\{x\alpha\} \subseteq A$ such that $L_{x\alpha} \xrightarrow{u} T$. Then for any $b \in A$, $bx\alpha \longrightarrow T(b)$, and so $\{bx\alpha\}$ is a Cauchy net in X. By hypothesis on A, $\|\mu_l(x)\|_q = \|L_x\|_q = q(x)$ and so μ_l is an isometry. Then it easily follows that $\{x\alpha\}$ is a Cauchy net in A. Since A is complete, $x\alpha \longrightarrow x \in A$. Hence, by continuity of μ_l,

$$T = \lim_{\alpha} \mu_l(x\alpha) = \mu_l(x) = L_x,$$

and so $T \in \mu_l(A)$. Thus $\mu_l(A)$ is u-closed in $M_l(A)$. ■

Theorem 4.1.4. *Suppose A is an F-algebra with a uniformly bounded left approximate identity. Then*

(a) $(M_l(A), s)$ is complete.

(b) $\mu_l(A)$ is s-dense in $M_l(A)$.

Proof. (a) Let $\{T\alpha : \alpha \in J\}$ be a Cauchy net in $(M_l(A), s)$. Then, for any $a \in A$, $\{T\alpha(a)\}$ is a Cauchy net in A. Consequently, the mapping $T : A \longrightarrow A$ given by $T(a) = \lim_{\alpha} T\alpha(a)$ $(a \in A)$ is well-defined and linear. Further, for any $a, b \in A$,

$$T(ab) = \lim_{\alpha} T\alpha(ab) = \lim_{\alpha} T\alpha(a)b = T(a)b,$$

so $T \in M_l(A)$. To show that $T\alpha \xrightarrow{s} T$, let D be a finite subset of A and $G \in \mathbb{W}$. There exists an index $\alpha_0 \in J$ such that

$$T\alpha(a) - T(a) \in G \quad \text{for all } \alpha \geq \alpha_o \text{ and all } a \in D.$$

Hence $T\alpha - T \in N(D, G)$ for all $\alpha \geq \alpha_o$. Thus $(M_l(A), s)$ is complete.

(b) Let $\{e\lambda : \lambda \in I\}$ be a left approximate identity for A. We need to show that $\mu_l(A)$ is s-dense in $M_l(A)$. Let $T \in M_l(A)$, and let D be a finite subset of A and $G \in \mathbb{W}$. We claim that, for some $\lambda \in I$, $\mu_l(T(e\lambda)) - T \in N(D, G)$. Now, by definition, $e\lambda a \longrightarrow a$ for all $a \in A$. Since D is finite and $T : A \longrightarrow A$ is continuous, we can choose a $\lambda_o \in I$ such that

$$T(ae\lambda) - T(a) \in G \quad \text{for all } a \in D \text{ and all } \lambda \geq \lambda_o.$$

Then, for any $a \in D$ and $\lambda \geq \lambda_o$,

$$L_{T(e\lambda)}(a) - T(a) = aT(e\lambda) - T(a) = T(ae\lambda) - T(a) \in G.$$

Thus $\mu_l(A)$ is s-dense in $M_l(A)$ and this completes the proof. ■

4.2 Characterization of Multipliers on Commutative Fréchet Algebras

In this section we present extensions of some results of Chapter 3 to topological algebra setting, due to Husain [12, 13] in a refined form..

Let A be a semisimple commutative topological algebra and $ML(A)$ the set of all linear (not necessarily continuous) multipliers on A.

Note that $M(A) = ML(A)$ if A is faithful or in particular has identity. Let $\Delta(A)$ denote the set of all non-zero continuous multiplicative linear functionals on A endowed with the relative w^*−topology $\sigma(A', A)$ of A'. In general, $\Delta(A)$ may be empty even for commutative Banach algebras. In the sequel, we shall always assume that $\Delta(A)$ is non-empty . For each $x \in A$, the Gelfand transform $\hat{x}$: $\Delta(A) \to \mathbb{C}$ is defined by

$$\hat{x}\,(f) = f(x), \quad f \in \Delta(A).$$

Theorem 4.2.1. *For each $T \in ML(A)$, there corresponds a continuous map $\phi = \phi^T : \Delta(A) \to \mathbb{C}$ such that*

$$\widehat{T(x)} = \phi.\,\hat{x} \text{ (pointwise)}, \; x \in A.$$

(that is, $\widehat{T(x)}(f) = \phi(f)\,\hat{x}\,(f), \;\; x \in A$ and $f \in \Delta(A)$).

Proof. We define $\phi = \phi^T : \Delta(A) \to \mathbb{C}$ by

$$\phi(f) = \phi^T(f) = f(T(x)), \; f \in \Delta(A).$$

where $x \in A$ is such that $f(x) = 1$. (Since $f \neq 0$,choose $y \neq 0$ in A with $f(y) \neq 0$; take $x = \frac{y}{f(y)}$). To see that ϕ is well-defined, let $x, y \in A$ with $f(x) = 1 = f(y)$. Then

$$f(T(xy)) = f(xT(y)) = f(x)f(T(y)) = f(T(y));$$

also

$$f(T(xy)) = f(T(yx)) = f(yT(x)) = f(y)f(T(x)) = f(T(x)).$$

Hence

$$\phi(f) = f(T(x)) = f(T(y)),$$

and so ϕ is independent of x.

We next show that, for any $x \in A$, (1) holds. Let $f \in \Delta(A)$. Fix $x_0 \in A$ such that $f(x_0) = 1$. Choose $\alpha, \beta \in \mathbb{C}, \beta \neq 0$, so that $f(\alpha x_0 + \beta x) = 1$.[Since

$$f(\alpha x_0 + \beta x) = 1 \Leftrightarrow \alpha f(x_0) + \beta f(x) = 1 \Leftrightarrow \alpha + \beta f(x) = 1,$$

we can take $\alpha = 1$ and any $\beta \neq 0$ if $f(x) = 0$, and $\alpha = 0$ and $\beta = \frac{1}{f(x)}$ if $f(x) \neq 0$.] Then, by the definition of ϕ,

$$f(T(\alpha x_0 + \beta x)) = \phi(f) = f(\alpha x_0 + \beta x)\phi(f).$$

Using linearity of T and f, we have

$$\begin{aligned}\alpha f(T(x_0)) + \beta f(T(x)) &= [\alpha f(x_0) + \beta f(x)]\phi(f) \\ &= \alpha f(x_0)\phi(f) + \beta f(x)\phi(f) \\ &= \alpha f(T(x_0)) + \beta f(x)\phi(f),\end{aligned}$$

or $\beta f(T(x)) = \beta f(x)\phi(f)$,or $f(T(x)) = f(x)\phi(f) = \phi(f)f(x)$,or

$$\widehat{T(x)}(f) = \phi(f)\, \hat{x}\,(f) = (\phi.\, \hat{x})(f).$$

We now prove that ϕ is continuous at any $f_0 \in \Delta(A)$. Let $0 < \varepsilon < 1$. Choose $x_0 \in A$ with $f_0(x_0) = 1$. Let $0 < \delta < 1$ whose value will be chosen later. Put $D = \{x_0, T(x_0)\}$, and let

$$U' = \{f \in \Delta(A) : |f(x) - f_0(x)| < \delta \text{ with } x \in D\}.$$

Clearly U' is a w*-neighbourhood of f_0 in $\Delta(A)$. Let $f \in U'$. Then $|f(x_0) - f_0(x_0)| < \delta$, or $f_0(x_0) - \delta < f(x_0) < f_0(x_0) + \delta$,or $0 < 1 - \delta < f(x_0) < 1 + \delta$, or $0 < \frac{1}{f(x_0)} < \frac{1}{1-\delta}$, and so, since $f(\frac{x_0}{f(x_0)}) = 1$,

$$\begin{aligned}|\phi(f) - \phi(f_0)| &= \left| f(T(\frac{x_0}{f(x_0)})) - f_0(T(x_0)) \right| \\ &= \frac{1}{|f(x_0)|} |f(T(x_0)) - f(x_0).f_0(T(x_0))| \\ &< \frac{1}{1-\delta}\{|f(T(x_0)) - f_0(T(x_0))| + |f_0(T(x_0)) - f(x_0)f_0(T(x_0))|\} \\ &= \frac{1}{1-\delta}\{\delta + |f_0(T(x_0))|\,|f_0(x_0) - f(x_0)|\} \\ &= \frac{1}{1-\delta}\{\delta + |h_0(T(x_0)).\delta|\} \\ &= \frac{\delta}{1-\delta}\{1 + |h_0(T(x_0))|\} \\ &= \varepsilon \quad \text{if } \delta = \frac{\varepsilon}{1 + |h_0(T(x_0))| + \varepsilon}\end{aligned}$$

Hence taking this value of δ in the definition of U', we have

$$|\phi(f) - \phi(f_0)| < \varepsilon \quad \text{for all } f \in U'.$$

Thus ϕ is continuous on $\Delta(A)$. This completes the proof. ■

Note. While proving the continuity of ϕ in the above Theorem, we may also take $D = \{x_0, T(x_0), x_0T(x_0)\}$ and $U' = \{f \in \Delta(A) : |f(x) - f_0(x)| < \delta \text{ with } x \in D\}$. Indeed, for any $f \in U'$,

$$\begin{aligned}
|\phi(f) - \phi(f_0)| &= \left|f(T(\frac{x_0}{f(x_0)})) - f_0(T(x_0))\right| \\
&\leq \left|\frac{f(T(x_0))}{f(x_0)} - f(T(x_0))\right| + |f(T(x_0)) - f_0(T(x_0))| \\
&< \frac{1}{|f(x_0)|}\,|f(T(x_0)) - f(x_0T(x_0))| + \delta \\
&< \frac{1}{1-\delta}\{|f(T(x_0)) - f_0(x_0T(x_0))| + |f_0(T(x_0)) - f(x_0T(x_0))|\} + \delta \\
&< \frac{1}{1-\delta}\{\delta + |f_0(x_0T(x_0)) - f(x_0T(x_0))|\} + \delta \\
&< \frac{1}{1-\delta}\{\delta + \delta\} + \delta = (\frac{2}{1-\delta} + 1)\delta \\
&= (\frac{3-\delta}{1-\delta})\delta \\
&< 4\delta \quad (\text{since } \delta < 1 \Leftrightarrow \frac{3-\delta}{1-\delta} < 4) \\
&= \varepsilon \ \text{ if } \delta = \frac{\varepsilon}{4.}
\end{aligned}$$

Hence taking $\delta = \frac{\varepsilon}{4}$ in the definition of U', we have

$$|\phi(f) - \phi(f_0)| < \varepsilon \quad \text{for all } f \in U'.$$

Thus ϕ is continuous on $\Delta(A)$. ■

Theorem 4.2.2 *Suppose A is faithful. Then the map $\Psi : T \to \phi^T$ of $M(A)$ into $C\ (\Delta(A))$ is an algebra homomorphism.*

Proof. Since A is faithful, $ML(A) = M(A)$. Hence, by the above theorem,

$$\phi^T \in C\ (\Delta(A)) \ \text{ for all } T \in M(A)$$

and so Ψ is well-defined. To show that Ψ is an algebra homomorphism, let $S, T \in M(A)$ and $\alpha, \beta \in \mathbb{C}$. Then $\alpha S + \beta T,\ ST \in M(A)$. Now, for any $f \in \Delta(A)$ and any $x \in A$ with $f(x) = 1$,

$$\begin{aligned}
\phi^{\alpha S+\beta T}(f) &= f[(\alpha S+\beta T)(x)] = f[\alpha S(x)+\beta T(x)] \\
&= \alpha f(S(x))+\beta f(T(x)) = \alpha\phi^{s}(f)+\beta\phi^{T}(f) \\
&= (\alpha\phi^{s}+\beta\phi^{T})f.
\end{aligned}$$

Hence $\phi^{\alpha S+\beta T} = \alpha\phi^{s}+\beta\phi^{T}$, or $\Psi(\alpha S+\beta T) = \alpha\Psi(S)+\beta\Psi(T)$, and so Ψ is linear.

Again, for any $f \in \Delta(A)$ and any $x \in A$ with $f(x) = 1$,

$$\begin{aligned}
\phi^{S}(f)\phi^{T}(f) &= f(S(x)f(T(x)) = f(S(x)T(x)) \\
&= f(xS(T(x))) = f(x)f((ST)(x)) \\
&= 1.\phi^{ST}(f) = \phi^{ST}(f).
\end{aligned}$$

Hence $\phi^{ST} = \phi^{S}\phi^{T}$,or $\Psi(ST) = \Psi(S)\Psi(T)$. Thus Ψ is an algebra homomorphism. ■

For Banach algebras, we can show (as in Chapter 3) that ϕ^{T} is also bounded on $\Delta(A)$ in the present setting.

Theorem 4.2.3. *Let A be a semisimple commutative Banach algebra. Then, for each $T \in M(A)$, ϕ^{T} is a bounded function on $\Delta(A)$ (i.e $\phi^{T} \in C_b(\Delta(A))$).*

Proof. Since A is a semisimple commutative Banach algebra, A is faithful, and so T is a bounded linear operator on A and so $\|T\| < \infty$. We claim that

$$\left\|\phi^{T}\right\|_{\infty} = \sup_{f\in\Delta(A)} \left|\phi^{T}(f)\right| \leq \|T\|.$$

Recall that, for each $f \in \Delta(A)$, $\|f\| \leq 1$. For each $f \in \Delta(A)$, let

$$K_f = \sup_{\|x\|=1} |f(x)| \leq 1.$$

By Theorem 4.2.1, for any $f \in \Delta(A)$ and $x \in A$,

$$\phi^{T}(f)\,\hat{x}\,(f) = \widehat{T(x)}\,(f)$$

or

$$\phi^{T}(f)f(x) = f(T(x)).$$

Therefore

$$\begin{aligned} \left|\phi^T(f)\right| |f(x)| &= |f(T(x))| \leq \|f\| \, \|T(x)\| \\ &\leq \|f\| \, \|T\| \, \|x\| \leq k_f \, \|T\| \, \|x\|, \end{aligned}$$

or

$$\begin{aligned} \left|\phi^T(f)\right| &\leq \frac{k_f \, \|T\| \, \|x\|}{|f(x)|} \quad \text{if } f(x) \neq 0. \\ &\leq \inf_{\|x\|=1} \frac{k_f \, \|T\| \, \|x\|}{|f(x)|} \\ &\leq \frac{k_f \, \|T\|}{\sup\limits_{\|x\|=1} |f(x)|} = \frac{k_f \, \|T\|}{k_f} = \|T\| < \infty. \end{aligned}$$

Hence $\left\|\phi^T\right\|_\infty < \infty$, and so ϕ^T is bounded on $\Delta(A)$. ■

Corollary 4.2.4. *Let A be a semisimple commutative Banach algebra. Then the map $h : T \to \phi^T$ of $M(A)$ into $C_b(\Delta(A))$ is norm-decreasing (i.e. $\|h(T)\|_\infty \leq \|T\|$).*

Proof. Let $T \in M(A)$. Then, as the proof of Theorem 4.2.3

$$\|h(T)\|_\infty = \left\|\phi^T\right\|_\infty = \sup_{f \in \Delta(A)} \left|\phi^T(f)\right| \leq \|T\|.$$

Hence h is norm-decreasing.

Theorem 4.2.5. *Let A be a semisimple commutative topological algebra such that $\Delta(A)$ separates the points of A. Then, for any linear mapping $T : A \to A$, the following are equivalent :*

(i) $T \in M(A)$

(ii) $T(M) \subseteq M$ for all maximal ideals $M \in \{ker(f) : f \in \Delta(A)\}$.

Proof. $(i) \Rightarrow (ii)$ Let $T \in M(A)$. Then, by Theorem 4.2.1, there exists a map $\phi^T : \Delta(A) \to \mathbb{C}$ such that

$$\phi^T(f) = f(T(x)), \quad f \in \Delta(A),$$

for some $x \in A$ with $f(x) = 1$. Let $M \in \{ker(f) : f \in \Delta(A)\}$.Then $M = M_f = ker(f)$ for a fixed $f \in \Delta(A)$. To show that $T(M_f) \subseteq M_f$, let $y \in M_f$. Let $x \notin M_f$. Since M_f is an ideal, $xy \in M_f$ and so $f(xy) = 0$. Now

$$f(x)f(T(y)) = f(xT(y)) = f(T(x)y)$$

$$= \quad f(T(x))f(y) = f(T(x)).0 = 0.$$

Since $x \notin M_f$, $f(x) \neq 0$ and so $f(T(y)) = 0$. Therefore $T(y) \in M_f$. Hence $T(M_f) \subseteq M_f$.

$(ii) \Rightarrow (i)$ Suppose $T(M_f) \subseteq M_f$ for all $f \in \Delta(A)$. To show that $T \in M(A)$, let $x, y \in A$. Consider arbitrary $f \in \Delta(A)$.

Case (I) Suppose $x \in M$ (or $y \in M_f$). Then $f(x) = 0$ (or $f(y) = 0$); also, since $T(M_f) \subseteq M_f$, $f(T(x)) = 0$ (or $f(T(y)) = 0$). Hence

$$\begin{aligned} f[xT(y) - T(x)y] &= f(xT(y)) - f(T(x)y) \\ &= f(x)f(T(y) - f(T(x))f(y) = 0 - 0 = 0. \end{aligned}$$

Case(II) Suppose $x \notin M_f$ and $y \notin M_f$. Since M_f is a maximal ideal, $\exists\ z \notin M_f$ such that $A = M_f \oplus [z]$. So $\exists\, u, v \in M_f$ and $\alpha, \beta \in \mathbb{K}$ such that $x = u + \alpha z$ and $y = v + \beta z$. Clearly $f(u) = 0$ and $f(v) = 0$. Also, since $T(M_f) \subseteq M_f$, $f(T(u)) = 0$ and $f(T(v)) = 0$. Hence, since T is linear,

$$\begin{aligned} f[xT(y) - T(x)y] &= f[(u + \alpha z)(T(v) + \beta T(z)) - (T(u) + \alpha T(z)(v + \beta z)] \\ &= f[(uT(v) - T(u)v) + \alpha(zT(v) - T(z)u) + \beta(uT(z) - T(u)z)] \\ &= 0 \end{aligned}$$

(since $f(u) = f(v) = f(T(u)) = f(T(v)) = 0$).

Thus, in each case,

$$f[xT(y) - T(x)y] = 0 \text{ for all f} \in \Delta(A).$$

Since $\Delta(A)$ separates the points of A,

$$xT(y) - T(x)y = 0, \text{ or } xT(y) = T(x)y.$$

This proves that $T \in M(A)$. ■

For further use, we restate Corollary 4.2.4.

Corollary 4.2.6. *Let A be a semisimple commutative Banach algebra. Then for each* $T \in M(A)$,

(i) $\phi^T(f) = f \circ T, f \in \Delta(A)$;

(ii) $\|\phi^T\|_\infty \leq \|T\|$.

Proof. This follows from Theorem 4.2.1 and Corollary 4.2.4. ■

4.3. Multipliers on Topological Algebras with Orthogonal Bases

In this section, we give characterizations of the multiplier algebra $M(A)$, due to Husain [12, 13]. It is shown that if A is a topological algebra with orthogonal Schauder basis, $M(A)$ can be continuously embedded into the algebra s of sequences of complex numbers; if, in addition, A has an identity and an unconditional orthogonal basis, then $M(A) \cong l_\infty$.

Let A be a topological algebra possessing an orthogonal basis $\{x_n\}$ with $\{f_n\}$ the associated sequence of coefficient functionals. Then we know from previous results that

(1) A is commutative.

(2) A is faithful.

(3) $\{x_n\}$ is a Schauder basis i.e. each f_n is continuous.

(4) $\Delta(A) = \{f_n\}$, which is a countable discrete space homeomorphic to the discrete space $\mathbb{N}$.

(5) A is semisimple.

(6) $\Delta(A)$ is point-separating.

Notation. Let s denote the algebra of all complex sequences with pointwise operations. Clearly $s = c(\mathbb{N})$, where $\mathbb{N}$ has the discrete topology. Note that $\ell_\infty \subseteq s$ and $\ell_\infty = C_b(\mathbb{N})$.

Theorem 4.3.1. Each $T \in M(A)$ is determined by a sequence $\{\phi^T\{f_n)\} \in s$.

Proof. By Theorem 4.2.1, $\phi^T : \Delta(A) \to \mathbb{C}$ is a continuous map given by

$$\phi^T(f) = f(T(x)),\ f \in \Delta(A),\ \text{where } x \in A \text{ with } f(x) = 1,$$

or

$$\phi^T(f) = f(T(\frac{x}{f(x)}))\ \text{ for all }\ f \in \Delta(A) \text{ and } x \neq 0 \text{ in } A,$$

or

$$f(T(x)) = \phi^T(f)f(x) \quad \text{for all }\ f \in \Delta(A)\ \text{ and } x \in A.$$

Since $\Delta(A) = \{f_n\}$, we have

$$f_i(T(x) = \phi^T(f_i)f_i(x)\ \text{ for all } i \geq 1 \text{ and } x \in A.$$

Since $f_i(x_i) = 1$, we have

$$f_i(T(x_i)) = \phi^T(f_i)f_i(x_i) = \phi^T(f_i), \quad i = 1, 2, ...$$

So, if $x = \sum_{i=1}^{\infty} f_i(x)x_i \in A$, then

$$\begin{aligned} T(x) &= \sum_{i=1}^{\infty} f_i(T(x)x_i \\ &= \sum_{i=1}^{\infty} \phi^T(f_i)f_i(x)x_i \quad \text{for all } x \in A. \end{aligned}$$

This shows that each $T \in M(A)$ is given by a sequence

$$\phi^T = \{\phi^T(f_i)\} \in s.$$

■

In view of the above theorem, we make the following

Definition 4.3.2. (1) Define a map $g : A \to s$ by

$$g(x) = \{f_i(x)\} \quad \text{for all } x = \sum_{i=1}^{\infty} f_i(x)x_i \in A.$$

Cleary, $g(T(x)) = \{f_i(Tx)\} = \{\phi^T(f_i)\}$.

(2) Define a map $h : M(A) \to s$ by

$$h(T) = \phi^T = \{\phi^T(f_i)\}$$

Theorem 4.3.3. *Let A be a topological algebra with an orthogonal Schauder basis $\{x_n\}$, and let $g : A \to s$ and $h : M(A) \to s$ be as defined above. Then:*

(a) g is a one-one continuous homomorphism.

(b) h is also a one-one continuous homomorphism.

Proof. The proofs of (a) and (b) are similar, so we only prove (b).

(b) Since A is faithful, commutative and semisimple, $h : M(A) \to s$ is an algebra homomorphism:

$$h(\alpha S + \beta T) = \phi^{\alpha S + \beta T} = \alpha\phi^s + \beta\phi^T = \alpha h(s) + \beta h(T)$$

$$h(ST) = \phi^{ST} = \phi^S\phi^T = h(S)h(T)$$

To show that h is one-one, let $T_1, T_2 \in M(A)$ with $h(T_1) = h(T_2)$. Then, for any $n \geq 1$,

$$\{\phi^{T_1}(f_n)\} = \{\phi^{T_2}(f_n)\},$$

or

$$f_n(T_1(x)) = f_n(T_2(x)) \text{ for all } x \in A \text{ with } f_n(x) = 1.$$

or

$$f_n(T_1(\frac{x}{f_n(x)})) = f_n(T_2(\frac{x}{f_n(x)})) \text{ for all } x \in A \text{ with } f_n(x) \neq 0.$$

Since A is semisimple, $rad\ A = \bigcap_{n=1}^{\infty} f_n^{-1}(0) = \{0\}$ and so $T_1(x) = T_2(x)$ for all $x \in A$. Hence $T_1 = T_2$.

Finally, since each f_n is continuous and also ϕ^T is continuous, it follows that h is continuous. ■

Notation. We denote

$$m(A) = \{\phi^T : T \in M(A)\} = \{\{\phi^T(f_i)\} : T \in M(A)\}.$$

Clearly, $m(A) \subseteq s$. Further, $h : M(A) \to m(A)$ is the map given by $h(T) = \phi^T, T \in M(A)$. Clearly h is onto.

Theorem 4.3.4. *Let A be a Banach algebra with an orthogonal basis $\{x_n\}$. Then $m(A) \subseteq \ell_\infty$.*

Proof. By Theorem 4.2.3, for each $T \in M(A)$, $\phi^T \in C_b(\Delta(A))$. But, with the discrete topology, $\Delta(A) = \{f_n\} \cong \mathbb{N}$. Hence, for each $T \in M(A)$, $\phi^T \in C_b(\mathbb{N}) = \ell_\infty$. Thus $m(A) \subseteq \ell_\infty$. ■

We shall later see in Theorem 4.3.6 that under some additional conditions, $m(A) = \ell_\infty$.

Corollary 4.3.5. *$M(A)$ is algebraically isomorphic to $m(A)$.*

Proof. Consider $h : M(A) \to m(A)$ given by

$$h(T) = \phi^T,\ T \in M(A).$$

Then, as in Theorems 4.2.2 and 4.3.3, for any $S, T \in M(A)$ and $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} h(\alpha S + \beta T) &= \phi^{\alpha S + \beta T} = \alpha\phi^S + \beta\phi^T \\ &= \alpha h(s) + \beta h(T) \end{aligned}$$

and

$$h(ST) = \phi^{ST} = \phi^S \phi^T = h(s)h(T).$$

Hence h is an algebra homomorphism. Clearly h is onto. To show that h is one-one, let $T_1, T_2 \in M(A)$ with $h(T_1) = h(T_2)$. Then $\phi^{T_1} = \phi^{T_2}$ and so

$$\phi^{T_1}(f) = \phi^{T_2}(f) \text{ for all } f \in \Delta(A)$$

or

$$f(T_1(x)) = f(T_2(x)) \text{ for all } f \in \Delta(A) \text{ and } x \in A.$$

Since $\Delta(A)$ is point-separating

$$T_1(x) = T_2(x) \quad \text{for all } x \in A.$$

Hence $T_1 = T_2$, and so h is one-one. Thus $M(A)$ is algebraically isomorphic to $m(A)$. ■

Note If A is Banach algebra with an orthogonal basis, then $m(A) \subseteq \ell_\infty$.Therefore we can endow $m(A)$ with the induced sup-norm

$$\|\phi^T\|_\infty = \sup_{i \geq 1} |\phi^T(f_i)|$$

Theorem 4.3.6. *Let A be a B_0-algebra (i.e. a localy convex F-algebras) with identity e and possessing an othogonal basis $\{x_n\}$.Then*

$$\{x_n\} \text{ is unconditional } \Leftrightarrow \ell_\infty \subseteq m(A) \ (i.e.m(A) = \ell_\infty).$$

Proof. [$\Rightarrow$] Suppose $\{x_n\}$ is unconditional, and let $\{\alpha_n\} \in \ell_\infty$. Note that $e = \sum_{i=1}^{\infty} f_i(e)x_i = \sum_{i=1}^{\infty} x_i$ (since each $f_i(e) = 1$) and so $\sum_{i=1}^{\infty} x_i$ converges. Since the basis $\{x_n\}$ is unconditional and $\{\alpha_n\} \in \ell_\infty$, the series $\sum_{i=1}^{\infty} \alpha_i x_i$ converges in A (see Section 1.6). Let $x_0 = \sum_{i=1}^{\infty} \alpha_i x_i$. .For any $x = \sum_{j=1}^{\infty} f_j(x)x_j \in A$,

$$L_{x_0}(x) = x_0 x = (\sum_{i=1}^{\infty} \alpha_i x_i)(\sum_{j=1}^{\infty} f_j(x)x_j) = \sum_{j=1}^{\infty} \alpha_j f_j(x)x_j,$$

or equivalently

$$f_i(L_{x_0}(x)) = \sum_{j=1}^{\infty} \alpha_j f_j(x) f_i(x_j) = \alpha_i f_i(x) \text{ for all } i \geq 1.$$

Hence $\{\alpha_n\}$ corresponds to a multipilier $L_{x_0} \in M(A)$, and so $\{\alpha_n\} \in m(A)$. Thus $\ell_\infty \subseteq m(A)$.

[$\Leftarrow$] Suppose $\ell_\infty \subseteq m(A)$. To show that $\{x_n\}$ is unconditional, we need to show that, for any $\{\alpha_n\} \in \ell_\infty$ and $x = \sum_{i=1}^{\infty} f_i(x)x_i \in A$, the sereis $\sum_{i=1}^{\infty} \alpha_i f_i(x)x_i$ is convergent. Since $\ell_\infty \subseteq m(A)$,

$\{\alpha_n\} = \{ \phi^T(f_n)\}$ for some $T \in M(A)$. Then, as in Theorem 4.3.1

$$\begin{aligned} T(x) &= \sum_{i=1}^{\infty} f_i(T(x)x_i \\ &= \sum_{i=1}^{\infty} \phi^T(f_i) f_i(x)x_i \\ &= \sum_{i=1}^{\infty} \alpha_i f_i(x)x_i \end{aligned}$$

Hence $\sum_{i=1}^{\infty} \alpha_i f_i(x) x_i$ is convergent to $T(x)$ in A. Thus $\{x_n\}$ is unconditinal. ■

Theoram 4.3.7 *Let A be a Banach algebra with an orthoganal basis $\{x_n\}$. Then the mapping $h : M(A) \to m(A)$ is norm-decreasing i.e. $\|h(T)\|_\infty \leq \|T\|$ for all $T \in M(A)$.*

Proof. Let $T \in M(A)$. Then, as in the proof of Theorem 4.2.3,

$$|\phi^T(f)| \leq \|T\| \quad \text{for all } f \in \Delta(A).$$

Hence

$$\|h(T)\|_\infty = \|\phi^T\|_\infty = \sup_{i \geq 1} |\phi^T(f_i)| \leq \|T\|.$$

■

Corollary 4.3.8. *Let A be a function algebra (i.e.. a Banach algebra with $\left\|\hat{x}\right\|_\infty = \|x\|, x \in X$) having an orthogonal basis $\{x_n\}$. Then $M(A)$ and $m(A)$ are isometric (and hence $m(A)$ is the Banach algebra ℓ_∞ because the only such an algebra A is c_0).*

Proof. Let $T \in M(A)$. We need to show that $\|\phi^T\|_\infty = \|T\|$. By above theorem, $\|\phi^T\|_\infty \leq \|T\|$. Since A is a function algebra, we have,

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \|T(x)\| = \sup_{\|\hat{x}\|_\infty = 1} \left\|\widehat{T(x)}\right\|_\infty \\ &= \sup_{\|\hat{x}\|_\infty = 1} \left\|\phi^T . \hat{x}\right\|_\infty \leq \sup_{\|\hat{x}\|_\infty = 1} \|\phi^T\|_\infty \left\|\hat{x}\right\|_\infty = \|\phi^T\|_\infty . \end{aligned}$$

Thus $\|h(T)\|_\infty = \|\phi^T\|_\infty = \|T\|$, and so $h : M(A) \to m(A)$ is an isometry. ■

Notation.

$$\ell_\infty \otimes A = \{\sum_{i=1}^{\infty} a_i f_i(x) x_i : \{a_i\} \in \ell_\infty,\ x = \sum_{i=1}^{\infty} f_i(x) x_i \in A\}.$$

Corollay 4.3.9. *Let A be a Banach algebra with an orthogonal unconditional basis $\{x_n\}$. Then $\ell_\infty \otimes A \subseteq A$.*

Proof. Let $\{a_i\} \in \ell_\infty$ and $x = \sum_{i=1}^{\infty} f_i(x) x_i \in A$. Since $\{x_n\}$ is an unconditional basis, the series $\sum_{i=1}^{\infty} a_i f_i(x) x_i$ is convergent in A. Hence $\ell_\infty \otimes A \subseteq A$. ■

Remark. Let A be a Banach algebra with an unconditional, normal, orthogonal basis $\{x_n\}$ such that $\left\|\hat{x}\right\|_\infty = \|x\|$ (*i.e.* A is also a function algebra). Then $A \cong c_0$ and so $M(A) \cong m(A) \cong \ell_\infty$; that is $M(c_0) \cong \ell_\infty$.

Theorem 4.3.10. *Let A be a topological algebra with an orthogonal Schauder basis $\{x_n\}$.Then, for any linear mapping $T : A \to A$,*

$$T \in M(A) \Leftrightarrow T(M_i) \subseteq M_i \textit{ for each } i \geq 1,$$

where $M_i = f_i^{-1}(0)$.

Proof. Since A has orthogonal Schauder basis, we have seen earlier that A is commutative and semisimple with $\Delta(A)$ point-separating. Hence, by Theorem 4.2.10,

$$T \in M(A) \Leftrightarrow T(M_i) \subseteq M_i \quad \text{for all } i \geq 1.$$

■

Note. We know from "conversion of Frèchet spaces into Frèchet algebras " that each Banach space E with an unconditional normalized basis $\{x_n\}$ can be given a multiplication

$$xy = \sum_{i=1}^{\infty} f_i(x) f_i(y) x_i, \quad x, y \in E$$

and an equivalent norm

$$\|x\| = sup\{\sum_{i \in F} f_i(x) x_i : F \text{ a finite subset of } \{x_n\}\}$$

making E a Banach algebra having the given basis as an orthogonal basis. We now have a result in the reverse direction as follows.

Theorem 4.3.11. *Let A be a Banach algebra with an orthogonal basis $\{x_n\}$. Then there exists an equivalent norm and a new multiplication making A a Banach algebra with $\{x_n\}$ as its basis but not an orthogonal basis.*

Proof. Let $T \in M(A)$ such that $T(x_i) \neq x_i$ for at least one x_i (e.g. any $T \neq I$). Consider $\{\phi^T(f_i)\} \in m(A)$. Then, for each $x = \sum_{i=1}^{\infty} f_i(x) x_i \in A$,

$$f_i(T(x)) = \phi^T(f_i) f_i(x) \quad \text{for all } i \geq 1 \text{ and } x \in A.$$

We define a new multiplication

$$x \circ y = xT(y) \quad \text{for all } x, y \in A.$$

Then it is easy to check that $(A, \circ)$ is an algebra . [To verify the associative law, for instance we, have

$$\begin{aligned} x \circ (y \circ z) &= x \circ (yT(z)) = xT(y.T(z)) \\ &= xT(y)T(z) \\ &= (x \circ y)T(z) \\ &= (x \circ y) \circ z \end{aligned}$$

for all $x, y, z \in A$.]

Define a new norm $\|.\|'$on A by

$$\|x\|' = \|T\| \cdot \|x\|, \quad x \in A.$$

It is a norm as:

(N_1) $0 \leq \|x\|' < \infty$ for all $x \in A$ (since$\|T\| < \infty$).

(N_2) $\|x\|' = 0 \Leftrightarrow \|T\| \, \|x\| = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$ (since$T \neq 0$).

(N_3) $\|\alpha x\|' = \|T\| \, \|\alpha x\| = |\alpha| \, \|T\| \, \|x\| = |\alpha| \, \|x\|'$.

(N_4) $\|x + y\|' = \|T\| \, \|x + y\| \leq \|T\| \, (\|x\| + \|y\|) = \|x\|' + \|y\|'$.

The equality $\|x\|' = \|T\| \, \|x\|$ with $\|T\|$ a constant shows that $\| \, \|'$ and $\|.\|$ are equivalent on A and so A is $\|.\|'$ −complete too.

Next $\|.\|'$ is an algebra norm on $(A, \circ)$ since

$$\begin{aligned} \|x \circ y\|' &= \|T\| \, \|x \circ y\| = \|T\| \, \|xT(y)\| \\ &\leq \|T\| \, \|x\| \, \|T(y)\| \leq \|T\| \, \|x\| \, \|T\| \, \|y\| \\ &= \|x\|' \cdot \|y\|'. \end{aligned}$$

Thus $(A, \circ, \|.\|')$ is a Banach algebra.

Now , since $x = \sum_{i=1}^{\infty} f_i(x)x_i$ converges in $\|.\|$ −norm and hence in the equivalent $\|.\|'$ −norm, $\{x_n\}$ is a basis in the new Banach space $(A, \circ, \|.\|')$.

To see that $\{x_n\}$ is not orthogonal in $(A, \circ, \|.\|')$, we observe that

$$\begin{aligned} x_i \circ x_j &= x_i T x_j = T(x_i x_j) \\ &= T(\delta_{ij} x_i) = \delta_{ij} T(x_i) \\ &\neq \delta_{ij} x_i \text{ for at least one } x_i \end{aligned}$$

(by our choice of T). ■

Corollary 4.3.12. *Let A be a Banach algebra with an orthogonal basis $\{x_n\}$. Then for each $i \geq 1$,*

$$f_i T = \phi^T(f_i) \text{ for all } T \in M(A).$$

Proof. Since $\Delta(A) = \{f_n\}$, we have

$$\phi^T(f_i) = f_i T(x) \quad \text{for all } i \geq 1 \text{ and } T \in M(A),$$

where $x \in A$ with $f_i(x) = 1$. ■

Finally, we consider the problem : Given a topological algebra A with an orthogonal basis , does there exist a topological algebra B such that $A \approx M(B)$? Since $M(B)$ always contains the identity $I : B \to B$, we may assume that A has an identity. Moreover, since $M(B)$ is often complete, we may assume that A is complete. With these remarks we have :

Theorem 4.3.13. *Let A be a complete topological algebra with identity and possessing an unconditional orthogonal basis $\{x_n\}$. Then there exists a topological algebra B such that $A \approx M(B)$.*

Proof. Take $B = (A', \sigma(A', A))$. We first show that, for any $g \in A'$, $\{\alpha_i\} = \{g(x_i)\} \in \ell_1$.If

$$x = \sum_{i=1}^{\infty} f_i(x) x_i \in A,$$

then, by continuity of g,

$$\begin{aligned} g(x) &= g(\sum_{i=1}^{\infty} f_i(x) x_i) = \sum_{i=1}^{\infty} f_i(x) g(x_i) \\ &= \sum_{i=1}^{\infty} f_i(x) \alpha_i. \end{aligned}$$

Since g is continuous, it is *spectrally continuous* which means that g, restricted to ℓ_∞ as a subspace of A, is continuous. (Note that the basis $\{x_n\}$ being unconditional implies for each $\{\alpha_n\} \in \ell_\infty$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$; that is, ℓ_∞ is embedded in A under the map $\{\alpha_n\} \to \sum_{i=1} \alpha_i x_i$). Thus $\{\alpha_i\} \in \ell_1$.

Now if $g \in A'$ is fixed and $x = \sum_{i=1}^{\infty} f_i(x)x_i \in A$, if $\epsilon_i = sgn\ (\alpha_i f_i(x))$, then $y = \sum_{i=1}^{\infty} \beta_i x_i \in A$; hence $\epsilon_i = f_i(y)$. So

$$\begin{aligned} |\alpha_i f_i(x)| &= \epsilon_i \alpha_i f_i(x) = f_i(y)\alpha_i f_i(x) \\ &= \alpha_i f_i(x) f_i(y) = \alpha_i f_i(xy), \end{aligned}$$

Therefore

$$g(xy) = \sum_{i=1}^{\infty} \alpha_i f_i(x) f_i(y) = \sum_{i=1}^{\infty} |\alpha_i f_i(x)| < \infty$$

implies that $\{\alpha_i f_i(x)\} \in \ell_1$. (We note that A' cannot have identity. But this is not needed in the proof.)

To show that A' is an algebra, let $g, h \in A'$ and define

$$(g.h)(x) = \sum_{i=1}^{\infty} g(x_i)h(x_i)f_i(x)$$

for $x = \sum_{i=1}^{\infty} f_i(x)x_i \in A$. Since $\{f(x_i)\} \in \ell_1$ and $\{g(x_i)f_i(x)\} \in \ell_1$, as shown above, the function $g.h : A \to \mathbb{K}$ is a well-defined continuous linear functional on A ; that is $g.h \in A'$. Further $g.h$ can be expressed as a composition of maps

$$x \to \{f_i \alpha_i\} \to \sum_{i=1}^{\infty} \alpha_i \beta_i f_i,$$

where $\alpha_i = g(x_i),\ \beta_i = h(x_i),\ f_i = f_i(x)$. Thus the map $(g, h) \to g.h$ is jointly continuous; i.e. B is a Hausdorff topological algebra

We now show that $A \approx M(B)$. Clearly $A \subseteq M(B)$. Under the above composition map embedding, to show that $M(B) \subseteq A$,we need to show that $m(B) \subseteq A$. Let $\alpha = \{\alpha_i\} \in m(B)$. If $x = \sum_{i=1}^{\infty} f_i(x)x_i$, then the map $g \to \sum_{i=1}^{\infty} \alpha_i g(x_i) f_i(x)$ of $B \to \mathbb{C}$ is a continuous linear functional and so $\alpha \in A''$. Since A' is equipped with the w$^*-$topology and A is complete, we have $A'' = A$ and so $\alpha \in A$. Therefore $m(B) \subseteq A$. Thus $A \approx M(B)$.

By the definition of product in A', $\{f_n\}$ is an orthogonal basis of A'. ■

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